Name___________ ~~No calculators~~. Present neatly. Score________.

1)

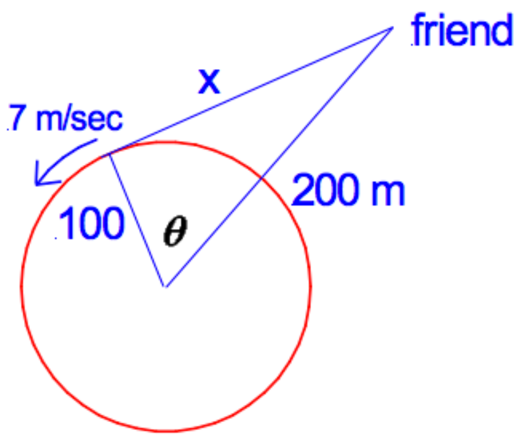
A runner sprints around a circular track of radius 100 m at a constant speed of 7 m/s. The runner's friend is standing at a distance 200 m from the center of the track. How fast is the distance between the friends changing when the distance between them is 200 m?

2)

A plane flying with a constant speed of 300 km/h passes over a ground radar station at an altitude of 1 km and climbs at an angle of 30°. At what rate is the distance from the plane to the radar station increasing a minute later?

Your work:

1)



Using the Law of Cosines,

$$x^2 = 100^2 + 200^2 - 2(100)(200)\cos\theta$$

The unknown in this problem is $\frac{dx}{dt}$.

Differentiate with respect to t

$$2x\frac{dx}{dt} = -40,000(-\sin\theta)\frac{d\theta}{dt}$$

$$\frac{dx}{dt} = \frac{40,000\sin\theta}{2x}\frac{d\theta}{dt} = \frac{20,000\sin\theta}{x}\frac{d\theta}{dt}$$

At the instant we want, $x = 200$ m.

We need $\frac{d\theta}{dt}$. We must convert 7 m/sec to rad/sec.

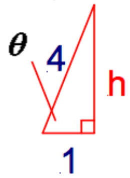
We use the fact that 1 complete revolution is $2\pi r = 2\pi(100) = 200\pi\, m$ and 1 revolution is 2π radians.

$$\frac{7\, m}{sec}\cdot\frac{1\, revolution}{200\,\pi\, m}\cdot\frac{2\pi\, rad}{1\, revolution} = \frac{7}{100}\frac{rad}{sec}$$

So at the instant we want, $\frac{d\theta}{dt} = \frac{7}{100}$ rad/sec .

We also need $sin\,\theta$. We can get $cos\,\theta$.
$x^2 = 100^2 + 200^2 - 40,000\,cos\,\theta \;\Rightarrow$
$200^2 = 100^2 + 200^2 - 40,000\,cos\,\theta \;\Rightarrow$
$cos\,\theta = \frac{10,000}{40,000} = \frac{1}{4}$.

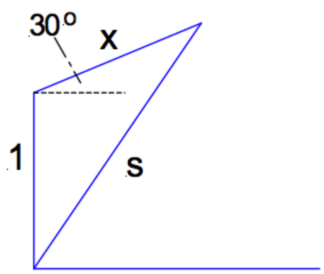
To find $sin\,\theta$, consider the right triangle below.



$h = \sqrt{16-1} = \sqrt{15} \;\Rightarrow\; sin\,\theta = \frac{\sqrt{15}}{4}$.

$\frac{dx}{dt} = \frac{20,000\,sin\,\theta}{x}\frac{d\theta}{dt} = \frac{20,000}{200}\cdot\frac{\sqrt{15}}{4}\cdot\frac{7}{100} = \frac{7\sqrt{15}}{4}$ m/sec

2)



The unknown in this problem is $\frac{ds}{dt}$.
We know the speed of the plane which is $\frac{dx}{dt} = 300$ km/hr .

The obtuse angle in the triangle is 120°, and we can use the Law of Cosines.
$s^2 = x^2 + 1^2 - 2(1)(x)\,cos\,120°$
$s^2 = x^2 + 1 - 2x(-1/2)$
$s^2 = x^2 + 1 + x$
Differentiate this equation with respect to t.
$2ss' = 2xx' + x'$
$s' = \frac{2xx' + x'}{2s}$.

We know $x' = 300$ and we can find x and s at the instant of time we want which is 1 minute later. Remember that 1 minute is 1/60 hr.

Since $D = RT$, $x = (300)(1/60) = 5$ miles.
$s^2 = x^2 + 1 + x \;\Rightarrow\; s^2 = (5)^2 + 1 + 5 = 31$
$\Rightarrow\; s = \sqrt{31}$ miles.

Therefore $s' = \frac{2xx' + x'}{2s} = \frac{2(5)(300) + 300}{2\sqrt{31}} =$

$\frac{3300}{2\sqrt{31}} = \frac{1650}{\sqrt{31}} \approx 296$ km/hr

Name____________ ~~No calculators~~. Present neatly. Score_________.

1)

The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o'clock?

2)

Two people start from the same point. One walks east at 3 mi/h and the other walks northeast at 2 mi/h. How fast is the distance between the people changing after 15 minutes?

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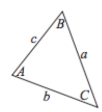
1)

This is a very tricky problem and will require the Law of Cosines:

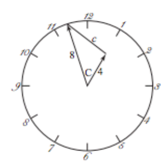


$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

for a triangle with standard labeling (see picture at right).

Okay, so let's get started!

Using the Law of Cosines, we can set up an equation involving c and C (see picture at right) as follows:



$$c^2 = 8^2 + 4^2 - 2(4)(8)\cos(C)$$

which simplifies to

$$c^2 = 80 - 64\cos(C) \qquad (1)$$

You can see that c, the distance between the tips of the hands, is mentioned in the problem (our goal is to find $\frac{dc}{dt}$), but it may not be obvious that we need the angle C. But the rotation of the hands of a clock is the only movement that affects the distance c, and we can measure that rate of rotation.

Taking the derivative of the above equation, we have

$$2c\,\frac{dc}{dt} = 64\sin(C)\,\frac{dC}{dt}. \qquad (2)$$

Now comes the fun part: at one o'clock, the angle C is $30°$, or $\frac{\pi}{6}$. Therefore $\sin(C) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$. Using the equation (1), we can also solve for what c is at that moment:

$$\begin{aligned} c^2 &= 80 - 64\cos\left(\frac{\pi}{6}\right) \\ &= 80 - 64\frac{\sqrt{3}}{2} = 80 - 32\sqrt{3}; \end{aligned}$$

therefore $c = \sqrt{80 - 32\sqrt{3}}$.

Finally, there is $\frac{dC}{dt}$. We need to decide what units we are going to use for time. Let's use hours (though minutes works just as well). The minute hand goes an angle of 2π radians per hour. The hour hand goes an angle of $\frac{\pi}{6}$ radians per hour in the same direction. So the angle between them changes at a rate of $2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$ radians per hour. At one o'clock, the angle between the hands is *decreasing* (look at or imagine a clock to visualize this), so we get

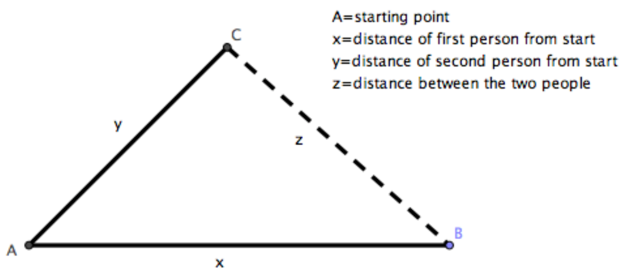
$$\frac{dC}{dt} = -\frac{11\pi}{6} \text{ radians per hour.}$$

Plugging all these numbers into the equation (2) and solving for $\frac{dc}{dt}$ we get

$$\begin{aligned} 2\sqrt{80 - 32\sqrt{3}}\,\frac{dc}{dt} &= 64 \cdot \frac{1}{2}\left(-\frac{11\pi}{6}\right) \\ \sqrt{80 - 32\sqrt{3}}\,\frac{dc}{dt} &= -\frac{88\pi}{3} \\ \frac{dc}{dt} &= -\frac{88\pi}{3\sqrt{80 - 32\sqrt{3}}} \approx -18.6 \text{ mm per hour} \end{aligned}$$

Thus the distance between the tips of the hands is decreasing at a rate of approximately $\boxed{\text{18.6 mm per hour}}$

2)



The Law of Cosines gives us that

$$z^2 = x^2 + y^2 - 2xy\cos\frac{\pi}{4}, \qquad \text{which is equivalent to} \qquad z^2 = x^2 + y^2 - \sqrt{2}\,xy.$$

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} - \left(\sqrt{2}\,x\frac{dy}{dt} + \sqrt{2}\,y\frac{dx}{dt}\right)$$

We are given that $\frac{dx}{dt} = 3$, $\frac{dy}{dt} = 2$ and after 15 minutes, we know that $x = 0.75$ miles and $y = 0.5$ miles. Plugging in these values of x and y into $z^2 = x^2 + y^2 - \sqrt{2}\,xy$, we can solve for $z = \frac{\sqrt{13-6\sqrt{2}}}{4}$ miles after 15 minutes.

Lastly, we plug all these values (which correspond to the moment in time when 15 minutes have passed) into our equation that relates the rates of change:

$$2\cdot\frac{\sqrt{13-6\sqrt{2}}}{4}\cdot\frac{dz}{dt} = 2\cdot 0.75\cdot 3 + 2\cdot 0.5\cdot 2 - \left(\sqrt{2}\cdot 0.75\cdot 2 + \sqrt{2}\cdot 0.5\cdot 3\right)$$

and solve for $\frac{dz}{dt}$.

$$\frac{dz}{dt} = \frac{(13-6\sqrt{2})}{\sqrt{13-6\sqrt{2}}} = \sqrt{13-6\sqrt{2}} \approx 2.1248$$

in miles per hour.