Quiz 48 | AP CALCULUS BC

1) $f(x) = \cos x \quad a = \frac{\pi}{3}$

n	0	1	2	3	...
$f^{(n)}(a)$	$1/2$	$-\sqrt{3}/2$	$-1/2$	$\frac{\sqrt{3}}{2}$	...

$$\cos x = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/3)}{n!}\left(x-\frac{\pi}{3}\right)^n$$

n even $\rightarrow \pm\cos\pi/3$

n odd $\rightarrow \pm\sin(\pi/3)$

$$= \frac{1}{2} - \frac{\sqrt{3}/2}{1!}\left(x-\frac{\pi}{3}\right) - \frac{1/2}{2!}\left(x-\frac{\pi}{3}\right)^2 + \dots$$

$$= \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 1/2}{(2n)!}\left(x-\frac{\pi}{3}\right)^{2n}}_{\text{evens}} + \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^{n+1} \cdot \sqrt{3}/2\,(x-\pi/3)^{2n+1}}{(2n+1)!}}_{\text{odds}}$$

2) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n(x-3)^n}{\sqrt{n+3}}$ Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}(x-3)^{n+1}}{\sqrt{n+4}} \cdot \frac{\sqrt{n+3}}{2^n \cdot (x-3)^n}\right| = \frac{2\sqrt{n+3}}{\sqrt{n+4}}|x-3| \rightarrow$

$\rightarrow 2|x-3| < 1 \iff |x-3| < 1/2 \iff -\frac{1}{2} < x-3 < \frac{1}{2} \iff \frac{5}{2} < x < \frac{7}{2}$

Radius $= 1/2$

Endpoints $x = \frac{5}{2}$: $\displaystyle\sum_{n=0}^{\infty} \frac{2^n \cdot (-1/2)^n}{\sqrt{n+3}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$ Converges, by Alternating Series Test.

Interval of Convergence $\left[\frac{5}{2}, \frac{7}{2}\right)$

$x = \frac{7}{2}$: $\displaystyle\sum_{n=0}^{\infty} \frac{2^n \cdot (1/2)^n}{\sqrt{n+3}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+3}}$ It diverges / Limit Comparison with $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}$ $(p = \frac{1}{2})$

3) "If $\displaystyle\sum_{n=0}^{\infty} a_n$ is absolutely convergent, then $\sum \underbrace{\frac{n+1}{n} a_n}_{b_n}$ is also absolutely convergent."

Ratio Test: $\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{n+2}{n+1} a_{n+1} \cdot \frac{n}{(n+1) \cdot a_n}\right| = \left|\underbrace{\frac{n(n+2)}{(n+1)^2}}_{\to 1} \cdot \underbrace{\frac{a_{n+1}}{a_n}}_{< 1}\right| \longrightarrow L < 1$ By Ratio Test on $\sum a_n$

4) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+1} \Rightarrow$ Alternating Series Test $\quad f(x) = \frac{\sqrt{x}}{x+1}$

$$f'(x) = \frac{(x+1)\cdot\frac{1}{2}x^{-1/2} - x^{1/2}}{(x+1)^2}$$

I. $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = \lim_{n\to\infty} \frac{1}{\frac{n}{\sqrt{n}} + \frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{\sqrt{n} + \frac{1}{\sqrt{n}}} = 0$

$$f'(x) = \frac{1}{2\sqrt{x}} \frac{[x+1-2x]}{(x+1)^2} = \frac{(1-x)}{2\sqrt{x}(x+1)^2}$$

By the Alternating Series Test, the series converges.

$f'(x) < 0$ for $x > 1$.

5) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\pi^n}{3^{2n}(2n)!} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{\pi}{9}\right)^n \Big/ (2n)! = \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}}_{\text{Cosine!}}$ for $x = \sqrt{\pi/9} = \frac{\sqrt{\pi}}{3}$.

So the series converges to $\cos\left(\frac{\sqrt{\pi}}{3}\right)$.

6) $\displaystyle\int \frac{\tan^{-1}x}{x}\,dx$

$$\tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \quad |x| < 1 \rightsquigarrow \frac{\tan^{-1}x}{x} = \frac{1}{x} \cdot \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{2n+1}$$

$$\rightsquigarrow \int \frac{\tan^{-1}x}{x}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)(2n+1)} = C + \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)^2} =$$

$$= C + \frac{x}{1^2} + \frac{(-1)^1 \cdot x^3}{3^2} + \frac{(-1)^2 \cdot x^5}{5^2} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)^2} + \dots$$

$$= C + \frac{x}{1^2} - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \frac{x^7}{7^2} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)^2} + \dots$$

Quiz 48 AP CALCULUS BC | SHUBLEKA

(1) $f(x) = \sin x \quad a = \pi/6$

n	0	1	2	3	etc.
$f^{(n)}(a)$	$1/2$	$\sqrt{3}/2$	$-1/2$	$-\frac{\sqrt{3}}{2}$	$\vdots$ ...

n = even $\quad (-1)^n \cdot \frac{1}{2}$

n = odd $\quad (-1)^n \cdot \frac{\sqrt{3}}{2}$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/6) \cdot (x-\pi/6)^n}{n!}$$

$$= \frac{1}{2} + \frac{\sqrt{3}/2}{1!}\left(x-\frac{\pi}{6}\right) + \frac{(-1/2)}{2!}\left(x-\pi/6\right)^2 + \dots$$

$$= \sum_{n=0}^{\infty}\left[\frac{\frac{1}{2}(-1)^n \cdot (x-\pi/6)^{2n}}{(2n)!} + \frac{\sqrt{3}}{2}\frac{(-1)^n \cdot \left(x-\frac{\pi}{6}\right)^{2n+1}}{(2n+1)!}\right]$$

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n \cdot (x-2)^n}{(n+2)!}$ Ratio Test: $\displaystyle\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}\cdot(x-2)^{n+1}}{(n+3)!}\cdot\frac{(n+2)!}{2^n\cdot(x-2)^n}\right| =$

$\displaystyle = \frac{2}{n+3}\cdot|x-2| \longrightarrow 0 < 1 \quad (-\infty, \infty) \quad R = \infty.$

(3) See other section's solutions.

(4) $$\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n-1}}{n} = \sum_{n=1}^{\infty}\frac{(\sqrt{n+1}-\sqrt{n-1})(\sqrt{n+1}+\sqrt{n-1})}{n(\sqrt{n+1}+\sqrt{n-1})} = \sum_{n=1}\frac{(n+1)-(n-1)}{n(\sqrt{n+1}+\sqrt{n-1})}$$

$$= \sum_{n=1}^{\infty}\frac{2}{n(\sqrt{n+1}+\sqrt{n-1})}$$

Limit Comparison with $\displaystyle\sum\frac{2}{n(\sqrt{n}+\sqrt{n})} = \sum\frac{1}{n\sqrt{n}}$ $\displaystyle = \sum\frac{1}{n^{3/2}}$. $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = 1 > 0$ Therefore both series converge.

(5) $\displaystyle\sum_{n=1}^{\infty}\tan^{-1}(n+1) - \tan^{-1}(n)$ Telescoping Series

$S_n = [\tan^{-1}(2) - \tan^{-1}(1)] + [\tan^{-1}3 - \tan^{-1}2] + \dots + [\tan^{-1}(n+1) - \tan^{-1}(n)]$

$S_n = -\tan^{-1}(1) + \tan^{-1}(n+1)$ $\displaystyle\lim_{n\to\infty} S_n = -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4}$. So, $\displaystyle\sum_{n=1}^{\infty}\tan^{-1}(n+1) - \tan^{-1}(n) = \frac{\pi}{4}$

(6) $f(x) = x^2 \cdot \tan^{-1}(x^3)$

$$\tan^{-1}(x^3) = \sum_{n=0}^{\infty}\frac{(-1)^n\cdot(x^3)^{2n+1}}{(2n+1)} = \sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^{6n+3}}{(2n+1)} \qquad |x^3| < 1 \iff |x| < 1$$

$$\Rightarrow x^2\cdot\tan^{-1}(x^3) = x^2\sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^{6n+3}}{(2n+1)} = \sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^{6n+5}}{(2n+1)}. \qquad \underline{R = 1.}$$