Mathematica Labs | Denis Shubleka

Subject: Calculus

Topic: Curvature; Tangent, Normal, and Binormal Vectors

Goal: Use *Mathematica* to compute the Tangent, Normal, and Binormal vectors on a vector-valued function r(t) at a point t = a. Define and compute curvature k(t) at a given point.

Task 1

In *Mathematica* we define and plot a vector valued function r(t), a helix in 3-space:

```
r[t_] := {Cos[t], Sin[t], t};

ParametricPlot3D[r[t], {t, 0, 10}]
```

The unit tangent vector T(t) is defined as $\frac{r'(t)}{|r'(t)|}$. Below we compute the unit tangent vector as a general rule, and then find a specific vector at t = $\pi/2$:

```
T[t_] := r'[t] / Norm[r'[t]];
T[Pi / 2]
```

The unit normal vector N(t) is defined as $\frac{T'(t)}{|T'(t)|}$. Below we compute the unit normal vector as a general rule Nrml[t], and then find a specific vector t = $\pi/2$:

```
Nrml[t_] := T'[t] / Norm[T'[t]];
Nrml[Pi / 2]
{0, -1, 0}
```

Note that we could also compute the normal vector directly:

```
T'[Pi / 2] / Norm[T'[Pi / 2]]
```

Finally, the Binormal vector B(t) is defined as cross product T(t) × N(t). Below we define a rule for the Binormal Vector, and then use it to compute a specific vector BiNrml(t) at t = $\pi/2$.

```
BiNrml[t_] := Cross[T[t], Nrml[t]];
BiNrml[Pi / 2]
```

After evaluating the previous cell, confirm that the dot product of B(t) with T(t) of N(t) is zero, since the three vectors are mutually orthogonal:

```
(BiNrml[Pi / 2].T[Pi / 2] == 0) && (BiNrml[Pi / 2].Nrml[Pi / 2] == 0)
```

The symbol == simply checks if the two sides are equal and prints ‘true’ when they are, and ‘false’ otherwise. With the ‘&&’ symbol we connect two or more statements.

Task 2

Curvature is a measure of how fast the tangent vector T(t) changes direction. One of the definitions of curvature is k(t) = |T′(t)|/|r′(t)|. Note that the range of k(t) consists of non-negative real numbers. Let’s investigate an example in 2-space. First, we define an ellipse and then compute curvature at t = 0 (far east corner) and t = $\pi/2$ (north pole).

```
r[t_] := {3 Cos[t], 4 Sin[t]};
```

```
T[t_] := (r'[t]) / Norm[r'[t]];
```

```
Norm[T'[0]] / Norm[r'[0]]
```

```
Norm[T'[Pi / 2]] / Norm[r'[Pi / 2]]
```

At which point is curvature the greatest? Does the computation agree with graph of the parametric curve?

```
ParametricPlot[r[t], {t, 0, 2 Pi}]
```

Related Exercises/Notes:

1. Use *Mathematica* to compute the vectors T(t), N(t), and B(t) at the given point:

$r(t) = <\cos t, \sin t, \ln\cos t>, (1, 0, 0)$