Mathematica Labs | Denis Shubleka

Subject: Calculus

Topic: Sequences and Series

Goal: Use *Mathematica* to explore the behavior of sequences and series.

Task 1

The Limit command helps us determine whether a sequences converges or diverges.
Find the infinity symbol in the Basic Math Assistant palette or simply type
Infinity. Below we look at a few sequences that you may have encountered in your
assigned reading or class notes. We'll start with an easy one:

```
Limit[1/n, n → Infinity]
```

Your turn: investigate the following sequences using *Mathematica*:

a) $\{n\cos(\frac{\pi}{n})\}$

b) $\{\frac{n!}{n^n}\}$

c) $\{(-1)^n\}$

d) $\left\{\frac{\cos(n)}{n^2}\right\}$

e) $\left\{\frac{(-1)^n}{n^2}\right\}$

It may be helpful to also plot the sequence, to visually confirm *Mathematica's*
answers. Input and run the following:

```
DiscretePlot[1/n, {n, 1, 100}]
```

Task 2

An infinite series $\sum_{n=1}^{\infty} a_n$ is the sum the terms of a sequence $\{a_n\}$. When the sum
exists, we say that the series converges; otherwise, it diverges.

Let us ask *Mathematica* to test a few series. Try these, one at a time:

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$$

$$\sum_{n=1}^{\infty} (-1)^n$$

$$\sum_{n=1}^{\infty} \mathrm{Sin}[n]$$

The **Sum** command can also be used to work with series, especially when an added variable is involved. To get the sum of the geometric series:
$1 + x^3 + x^6 + x^9 + \ldots,$
enter and run the following:

$\mathrm{Sum}[x^n, \{n, 0, \infty, 3\}]$

Remember that the geometric series above will converge whenever the ratio is less
than 1 in absolute value. *Mathematica* does not tell us that $|x^3| < 1$, which is
equivalent to $|x| < 1$.

Next, we take a look at the **Series** command, which gives the Taylor Series of an
algebraic expression. Think of this command as the inverse process of **Sum.** The
example below gives the Taylor series representation of $\frac{1}{1-x}$, centered at $x = 0$ and
expanded up to degree 6.

$\mathrm{Series}\left[\frac{1}{1-x}, \{x, 0, 6\}\right]$

The big O term simply indicates that there are more
terms. Use the Normal command to eliminate the big O term from the final answer,
as shown below.

```
Normal[%]
```

$1 + x + x^2 + x^3 + x^4 + x^5 + x^6$

It is helpful to compare the original function and its Taylor Polynomial in the
same plot:

```
Plot[{%, 1/(1 - x)}, {x, -1, 1}]
```

Would the Taylor Polynomial $T_6(x)$ do a good approximation job for x values close
to 0?

Task 3

To get a general formula for the Taylor series expansion, type and execute:

```
Normal[Series[f[x], {x, a, 5}]]
```

Next, we define a general function **findTaylor** that helps us compute a Taylor polynomial for $f(x)$, centered at $x = x_0$ with degree n.

```
findTaylor[f_, {x_, x0_}, n_] := Normal[Series[f[x], {x, x0, n}]]
```

Let us put this operation to the test by determining a representation of two
common trig functions:

```
findTaylor[Sin, {x, 0}, 7]
```

Recall that 5!=120 and 7!=5040. The result should not be surprising. And here is
one more example:

```
findTaylor[Cos, {x, π/3}, 6]
```

Feel free to plot the Taylor Polynomial above and y = Cos[x] in the same window.

```
Plot[{%, Cos[x]}, {x, -3, 3}]
```

Related Exercises/Notes: