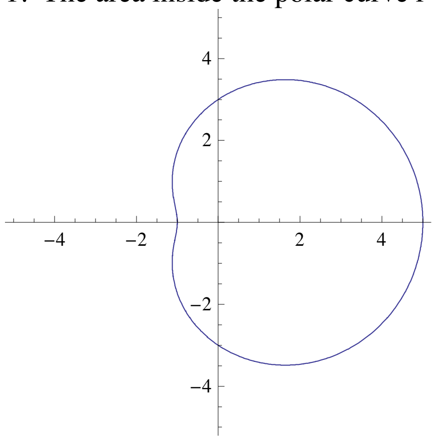
AP Calculus BC
Worksheet: Polar Coordinates

1. The area inside the polar curve r = 3 + 2cos θ is



(A) 9.425

(B) 18.850

(C) 28.274

(D) 34.558

(E) 69.115

2. The area enclosed inside the polar curve $r^2 = 10 \cos(2\theta)$ is

(A) 10

(B) 5π

(C) 20

(D) 10π

(E) 25π

3. The area enclosed by the polar curve r cos $\frac{1}{2}\theta = 1$ in the interval $0 \le \theta \le \frac{\pi}{2}$ is

(A) $\frac{1}{2}$

(B) $\frac{\sqrt{2}}{2}$

(C) $\frac{\pi}{4}$

(D) 1

(E) 2

4. What is the area enclosed by the lemniscate $r^2 = -25\cos 2\theta$?

(A) $\frac{25}{8}$

(B) $\frac{25}{4}$

(C) $\frac{25}{2}$

(D) 25

(E) 50

5. The area of the region inside the polar curve r = 4 sin θ and outside the polar curve r = 2 is given by

(A) $\frac{1}{2}\int_0^{\pi}(4\sin\theta - 2)^2\,d\theta$

(B) $\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}(4\sin\theta - 2)^2\,d\theta$

(C) $\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}(4\sin\theta - 2)^2\,d\theta$

(D) $\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\left(16\sin^2\theta - 4\right)d\theta$

(E) $\frac{1}{2}\int_0^{\pi}\left(16\sin^2\theta - 4\right)d\theta$

6. Which of the following is equal to the area of the region inside the polar curve r = 2 cos θ and outside the polar curve r = cos θ?

(A) $3\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta$

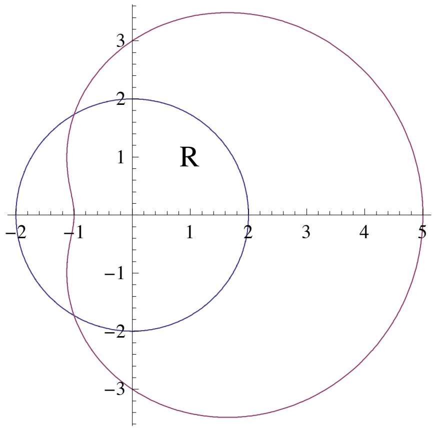
(B) $3\int_0^{\pi}\cos^2\theta\,d\theta$

(C) $\frac{3}{2}\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta$

(D) $3\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta$

(E) $3\int_0^{\pi}\cos\theta\,d\theta$

7. The graphs of the polar curves r = 2 and r = 3 + 2cos θ are shown in the figure below. The curves intersect when $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$.



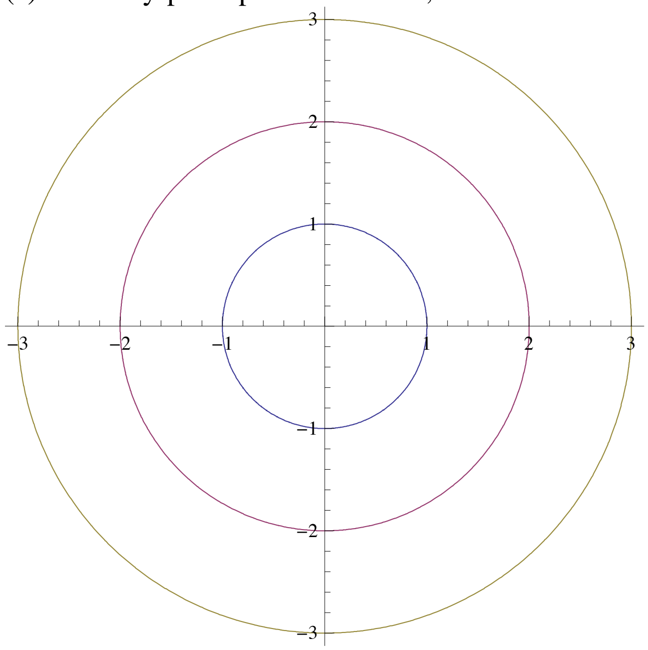
(a) Let R be the region that is inside the graph of r = 2 and also inside the graph of r = 3 + 2 cos θ, as indicated above. Find the area of R.

(b) A particle moving with nonzero velocity along the polar curve given by r = 3 + 2 cos θ has position (x(t), y(t)) at time t, with $\theta = 0$ when t = 0. This particle moves along the curve so that $\frac{dr}{dt} = \frac{dr}{d\theta}$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

(c) For the particle described in part (b), $\frac{dy}{dt} = \frac{dy}{d\theta}$. Find the value of $\frac{dy}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

8. Consider the polar curve $r = 2 \sin(3\theta)$ for $0 \leq \theta\, \pi$.

(a) In the xy-plane provided below, sketch the curve.



(b) Find the area of the region inside the curve.

(c) Find the slope of the curve at the point where $\theta = \frac{\pi}{4}$.

Polar Curves Worksheet: Solutions (prepared by D. Shubleka)

Problem 1 (Multiple Choice)

$r = 3 + 2\cos\theta$

$\theta = 0 \longrightarrow r = 5$

$\theta = \pi \longrightarrow r = -1$

Using Symmetry: $\text{Area} = 2\int_0^{\pi} \frac{1}{2}[r(\theta)]^2 \, d\theta$

$= \int_0^{\pi} [3 + 2\cos\theta]^2 \, d\theta$

$= \int_0^{\pi} 9 + 12\cos\theta + 4\cos^2\theta \, d\theta$

$= \int_0^{\pi} 9 + 12\cos\theta + 2\cos(2\theta) + 2 \, d\theta$

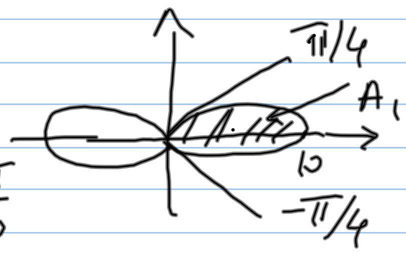
$= (11\theta + 12\sin\theta + \sin 2\theta)\Big|_0^{\pi} = 11\pi \approx 34.558$

$\boxed{D}$

Problem 2

Find the area enclosed inside the curve

$r^2 = 10 \cos 2\theta$



$r = 0 \Rightarrow \cos 2\theta = 0$

$2\theta = \frac{\pi}{2}$ or $-\frac{\pi}{2}$

$\theta = \frac{\pi}{4}$ or $\frac{-\pi}{4}$

$A_1 = 2 \cdot \int_0^{\pi/4} \frac{r^2}{2}\, d\theta = \int_0^{\pi/4} 10 \cos 2\theta \, d\theta$

$= 5 \sin 2\theta \Big|_0^{\pi/4} = 5$

Total Area $= 5 + 5 = 10$ $\boxed{A}$

Problem 3 (Multiple Choice)

The area enclosed by the polar curves

$r = \dfrac{1}{\cos[\frac{1}{2}\theta]}$ in the interval $[0, \frac{\pi}{2}]$

$r(\theta) = \sec[\frac{\theta}{2}]$

$$A = \int_0^{\pi/2} \frac{r^2(\theta)}{2}\, d\theta = \int_0^{\pi/2} \frac{1}{2} \sec^2\left(\frac{\theta}{2}\right) d\theta =$$

$$= \tan\left(\frac{\theta}{2}\right) \Big|_0^{\pi/2} = \tan(\pi/4) - \tan(0)$$

$$= 1$$

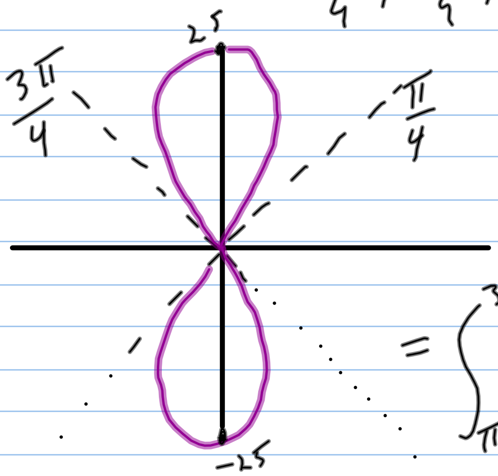
$$\boxed{D}$$

Problem 4 (Multiple Choice)

What is the enclosed area by the lemniscate

$r^2 = -25\cos 2\theta$?

$r=0 \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{-\pi}{4}, \frac{-3\pi}{4}$ etc.



$$A = 2\int_{\pi/4}^{3\pi/4} \frac{1}{2} r^2 \, d\theta =$$

$$= \int_{\pi/4}^{3\pi/4} -25\cos 2\theta \, d\theta = -\frac{25}{2}\sin 2\theta \Big|_{\pi/4}^{3\pi/4}$$

$$= -\frac{25}{2}(-1-1) = 25 \quad \boxed{D}$$

Problem 5 Multiple Choice

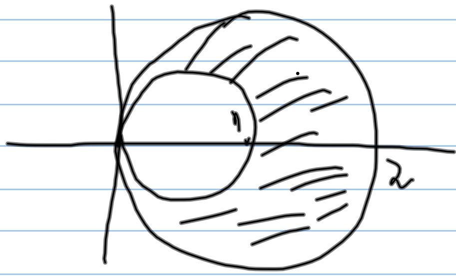
$$\left.\begin{array}{l} r = 4\sin\theta \\ r = 2 \end{array}\right\} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (4\sin\theta)^2 - 2^2 \, d\theta$$

$$\boxed{D}$$

Problem 6 Multiple Choice

Inside $r = 2\cos\theta$
Outside $r = \cos\theta$ $\Rightarrow \cos\theta = 0$



$$A = 2\int_0^{\pi/2} \frac{(2\cos\theta)^2}{2} - \frac{\cos^2\theta}{2}\, d\theta$$

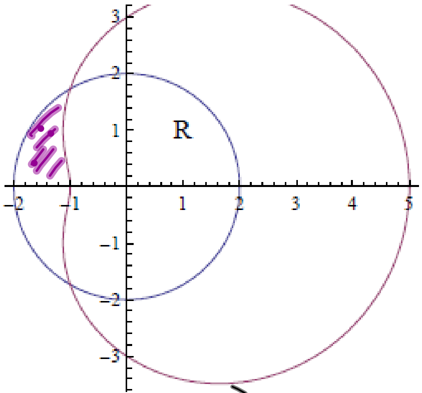
$$= \int_0^{\pi/2} 4\cos^2\theta - \cos^2\theta\, d\theta$$

$$= 3\int_0^{\pi/2} \cos^2\theta\, d\theta$$

$\boxed{A}$

Problem 7 Free Response.

a)



$$R = \pi r^2\Big|_{r=2} - 2\cdot[\text{purple}]$$

$$= 4\pi - 2[\text{purple}]$$

$$2[\text{purple}] = \left[\frac{1}{2}\int_{\frac{2\pi}{3}}^{\pi} 2^2 - [3+2\cos\theta]^2\, d\theta\right]\cdot 2$$

$$= \int_{\frac{2\pi}{3}}^{\pi} -5 - 12\cos\theta - 4\cos^2\theta\, d\theta = \left(-5\theta - 12\sin\theta\right)\Big|_{\frac{2\pi}{3}}^{\pi} - \int_{\frac{2\pi}{3}}^{\pi} 2\cos 2\theta + 2\, d\theta$$

$$= -5\theta - 12\sin\theta - \sin 2\theta - 2\theta\Big|_{2\pi/3}^{\pi}$$

$$= 7\theta + 12\sin\theta + \sin 2\theta\Big|_{\pi}^{2\pi/3} = \frac{14\pi}{3} + 6\sqrt{3} - \frac{\sqrt{3}}{2} - 7\pi =$$

$$= \frac{-7\pi}{3} + \frac{11\sqrt{3}}{2}$$

$$R = 4\pi - \left[\frac{-7\pi}{3} + \frac{11\sqrt{3}}{2}\right]$$

$$= \boxed{\frac{19\pi}{3} - \frac{11\sqrt{3}}{2}}$$

Problem 7 Part b)

$r = 3 + 2\cos\theta \qquad \frac{dr}{dt} = \frac{dr}{d\theta}$

$\frac{dr}{d\theta} = -2\sin\theta = \frac{dr}{dt}$ @ $\frac{\pi}{3}$ we have:

$$\left.\frac{dr}{dt}\right|_{\theta=\frac{\pi}{3}} = \left.\frac{dr}{d\theta}\right|_{\theta=\frac{\pi}{3}} = -2 \cdot \frac{\sqrt{3}}{2} = -\sqrt{3}. < 0$$

The particle is heading towards the pole when $\theta = \frac{\pi}{3}$. (Note $r(\frac{\pi}{3}) > 0$)

Problem 7 Part c

$$y = r \cdot \sin\theta = (3 + 2\cos\theta)\sin\theta$$
$$= 3\sin\theta + 2\sin\theta\cos\theta = 3\sin\theta + \sin 2\theta$$

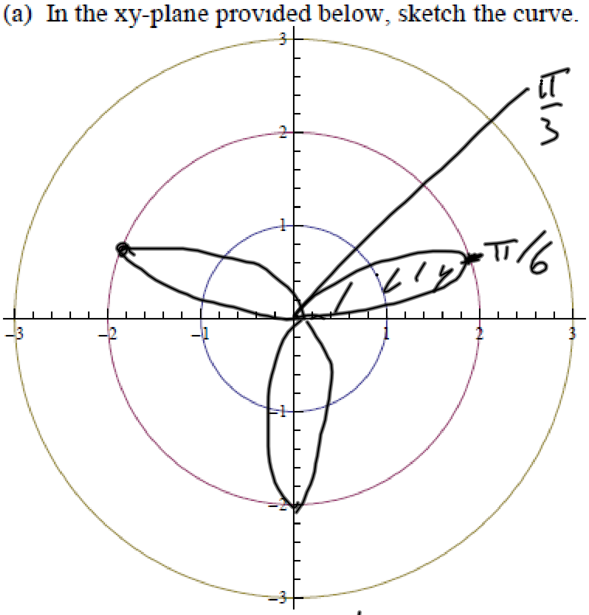
$$\left.\frac{dy}{dt}\right|_{\frac{\pi}{3}} = \left.\frac{dy}{d\theta}\right|_{\frac{\pi}{3}} = 3\cos\theta + (\cos 2\theta)\cdot 2\Big|_{\theta=\frac{\pi}{3}} =$$

$$= 3\cdot\frac{1}{2} + 2\cdot\cos\left(\frac{2\pi}{3}\right) = \frac{3}{2} - 1 = \frac{1}{2}.$$

The particle's y-coordinate is increasing. It is going up when $\theta = \frac{\pi}{3}$.

Problem 8

(a) In the xy-plane provided below, sketch the curve.



$r = 2\sin(3\theta)$

$0 \le \theta \le \pi$

$r = 0$

$3\theta = 0, \pi, 2\pi$

$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}$

b) $A = 3 \cdot \frac{1}{2} \int_0^{\pi/3} (2\sin 3\theta)^2 \, d\theta =$

$= \int_0^{\pi/3} 6\sin^2(3\theta)\, d\theta$

$= \int_0^{\pi/3} 6 \, \frac{1-\cos 6\theta}{2} \, d\theta = \frac{1}{2}\left(6\theta - \sin 6\theta\right)\Big|_0^{\pi/3}$

$= \frac{1}{2}(2\pi) = \boxed{\pi}$ Each petal has area $\pi/3$.

c) Find slope $\frac{dy}{dx}$ @ $\theta = \frac{\pi}{4}$

$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(2\sin 3\theta \cdot \sin\theta\right) = 6\cos 3\theta \sin\theta + 2\sin 3\theta\cos\theta \Big|_{\pi/4}$

$= 6\left(\frac{-\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2} + 2 \cdot \frac{\sqrt{2}}{2} \frac{\sqrt{2}}{2} = -2$

$\frac{dx}{d\theta} = \frac{d}{d\theta}\left(2\sin 3\theta \cos\theta\right) = 6\cos 3\theta\cos\theta + 2\sin 3\theta(-\sin\theta)\Big|_{\pi/4}$

$= 6\left(\frac{-\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + 2 \cdot \frac{\sqrt{2}}{2}\left(-\frac{\sqrt{2}}{2}\right)$

$= -3 - 1 = -4$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-2}{-4} = \frac{1}{2} = \text{slope}$