WORKSHEET: Series, Taylor Series

No Calculator Section

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Name:____________________________ **Date:** ________________ **Period:** _______

Part I. Multiple-Choice Questions (5 points each; please circle the correct answer.)

1. The series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n e^n}{n^2}$ diverges because

   I. The terms do not tend to 0 as n tends to ∞.

   II. The terms are not all positive.

   III. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$.

   (A) I only
   (B) II only
   (C) III only
   (D) I and II only
   (E) I and III only

2. The interval of convergence for the series $\sum_{n=1}^{\infty} \frac{(3x+2)^{n+1}}{n^{5/2}}$ is

   (A) $-1 \leq x < -\frac{1}{3}$
   (B) $-1 < x \leq -\frac{1}{3}$
   (C) $-1 \leq x \leq -\frac{1}{3}$
   (D) $\frac{1}{3} \leq x \leq 1$
   (E) $-1 < x < \frac{1}{3}$

3. Given that $f(x) = \sum_{n=0}^{\infty} \frac{n(x-a)^n}{2^n}$ on the interval of convergence of the Taylor series, $f^{(4)}(a) =$

   (A) 0
   (B) 6
   (C) 9
   (D) $\frac{1}{4}$
   (E) $\frac{1}{4!}$

4. Which of the following series converge?

I. $\sum_{n=1}^{\infty}\left(\frac{n^2-n+5}{n^{7/2}+1}\right)$.

II. $\sum_{n=1}^{\infty}\frac{(-1)^n 3}{n}$

III. $\sum_{n=1}^{\infty}\left(\frac{\cos 2n\pi}{n^2}\right)$.

(A) I and II only

(B) I and III only

(C) II and III only

(D) They all do!

(E) None of them do!

5. $1-\frac{\pi^2}{2!}+\frac{\pi^4}{4!}+\cdots+(-1)^n\frac{\pi^{2n}}{(2n)!}+\cdots=$

(A) 0

(B) -1

(C) π

(D) 1

(E) $-\pi$

Part II. Free-Response Questions

1. A function f is defined by

$$f(x) = \frac{1}{4} + \frac{2}{4^2}x + \frac{3}{4^3}x^2 + \cdots + \frac{n+1}{4^{n+1}}x^n + \cdots$$

for all x in the interval of convergence of the given power series.

(a) (**4 points**) Find the interval of convergence for this power series. Show the work that leads to your answer.

(b) (**3 points**) Find $\lim_{x \to 0} \dfrac{f(x) - \frac{1}{4}}{x}$.

#1, continued; $f(x) = \frac{1}{4} + \frac{2}{4^2}x + \frac{3}{4^3}x^2 + \cdots + \frac{n+1}{4^{n+1}}x^n + \cdots$

(c) (**3 points**) Write the first three nonzero terms and the general term for an infinite series that represents $\int_0^2 f(x)\,dx$.

(d) (**4 points**) Find the sum of the series determined in part (c).

2. Let f be a function with derivatives of all orders and for which $f(2) = 7$. When n is odd, the nth derivative of f at $x = 2$ is 0. When n is even and $n \geq 2$, the nth derivative of f at $x = 2$ is given by $f^{(n)}(2) = \dfrac{(n-1)!}{3^n}$.

   (a) (**4 points**) Write the sixth-degree Taylor polynomial for f about $x = 2$.

   (b) (**3 points**) In the Taylor series for f about $x = 2$, what is the coefficient of $(x-2)^{2n}$ for $n \geq 1$?

   (c) (**4 points**) Find the inteval of convergence of the Taylor series for f about $x = 2$. Show the work that leads to your answer.

Calculators Allowed

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Part III. Multiple-Choice Questions (5 points each; please circle the correct answer.)

1. The series $\sum_{n=1}^{\infty} \frac{\sqrt{n^p - 1}}{n^{p+2} + 1}$ will converge, provided that

   (A) $p > 1$
   (B) $p > 2$
   (C) $p > -1$
   (D) $p > -2$
   (E) $p > 0$

2. The graph of the function represented by the Taylor series $\sum_{n=0}^{\infty} n(x+1)^{n-1}$ intersects the graph of $y = \ln x$ at $x \approx$

   (A) 1.763
   (B) 0.703
   (C) 1.532
   (D) 0.567
   (E) 1.493

3. Using the fourth-degree Maclauren polynomial of the function $f(x) = e^x$ to estimate e^{-2}, this estimate is

   (A) 7.000
   (B) 0.333
   (C) 0.135
   (D) 0.067
   (E) 0.375

4. What is the approximation of the value of $\cos(2^\circ)$ obtained by using the sixth-degree Taylor polynomial about $x = 0$ for $\cos x$?

(A) $1 - 2 + \frac{2}{3} - \frac{4}{45}$

(B) $1 + 2 + \frac{16}{24} + \frac{64}{720}$.

(C) $1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720}$

(D) $1 - \frac{\pi^2}{2 \cdot 90^2} + \frac{\pi^4}{4! \cdot 90^4} - \frac{\pi^6}{6! \cdot 90^6}$

(E) $1 + \frac{\pi^2}{2 \cdot 90^2} + \frac{\pi^4}{4! \cdot 90^4} + \frac{\pi^6}{6! \cdot 90^6}$

5. Which of the following gives a Taylor polynomial approximation about $x = 0$ for $\sin 0.5$, correct to four decimal places?

(A) $0.5 + \frac{(0.5)^3}{3!} + \frac{(0.5)^5}{5!}$

(B) $0.5 - \frac{(0.5)^3}{3!} + \frac{(0.5)^5}{5!}$

(C) $0.5 - \frac{(0.5)^3}{3} + \frac{(0.5)^5}{5}$

(D) $0.5 + \frac{(0.5)^2}{2!} + \frac{(0.5)^3}{3!} + \frac{(0.5)^4}{4!} + \frac{(0.5)^5}{5!}$

(E) $0.5 - \frac{(0.5)^2}{2!} + \frac{(0.5)^3}{3!} - \frac{(0.5)^4}{4!} + \frac{(0.5)^5}{5!}$

Part IV. Free-Response Questions

1. The function f has derivatives of all orders for all real numbers x. Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.

   (a) (**3 points**) Write the third-degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1)$.

   (b) (**4 points**) The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \leq 3$ for all x in the closed interval $[1, 2]$. Use the Lagrange error bound on the approximation to $f(1)$ found in part (a) to explain why $f(1) \neq -5$.

   (c) (**4 points**) Write the fourth-degree Taylor polynomial, $P(x)$, for $g(x) = f(x^2 + 2)$ about $x = 0$. Use P to explain why g must have a relative minimum at $x = 0$.

2. Let f be a function having derivatives of all orders for al real numbers. The third-degree Taylor polynomial for f about $x = 2$ is given by

$$T(x) \;=\; -5(x-2)^2 - 3(x-2)^3.$$

(a) (**2 points**) Find $f(2)$ and $f''(2)$.

(b) (**4 points**) Is there enough information given to determine whether f has a critical point at $x = 2$? If not, explain why not. If so, determine whether $f(2)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

#2, continued; $T(x) = -5(x-2)^2 - 3(x-2)^3$.

(c) (**4 points**) Use $T(x)$ to find an approximation for $f(0)$. Is there enough information given to determine whether f has a critical point at $x = 0$? If not, explain why not. If so, determine whether $f(0)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

(d) (**4 points**) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \leq 5$ for all x in the closed interval $[0, 2]$. Use the Lagrange error bound on the approximation to $f(0)$ found in part (c) to explain why $f(0)$ is positive.