Given a function $f(x)$ that has positive $y-$values on a closed interval $[a,b]$, we measure the area that the graph of the function forms with the horizontal axis and the lines $x=a, x=b$ by evaluating the limit:

$$A=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

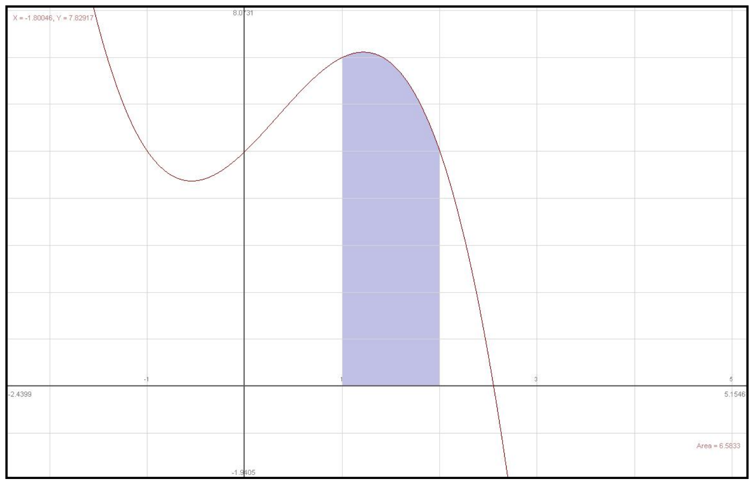


Figure: $f(x)=-x^3+x^2+2x+5 \quad [a,b]=[1,2]$

Translating the Information	Properties of Sigma
$x_0=a$ $x_n=b$ $\Delta x=\frac{b-a}{n}$ $x_1=x_0+\Delta x$ $x_2=x_0+2\Delta x$ .... $x_i=x_0+i\Delta x$ ... $x_n=x_0+n\Delta x=a+n\frac{b-a}{n}=a+b-a=b$	$\sum_{i=1}^{n} k=(\underbrace{k+k+....+k}_{n \text{ times}})=nk$ $\sum_{i=1}^{n}(x_i\pm y_i)=\sum_{i=1}^{n}x_i\pm\sum_{i=1}^{n}y_i$ $\sum_{i=1}^{n}kx_i=k\sum_{i=1}^{n}x_i$ $\sum_{i=1}^{n}i=1+2+3+...+n=\frac{n(n+1)}{2}$ $\sum_{i=1}^{n}i^2=1^2+2^2+3^2+...+n^2=\frac{n(n+1)(2n+1)}{6}$ $\sum_{i=1}^{n}i^3=1^3+2^3+3^3+...+n^3=\left(\frac{n(n+1)}{2}\right)^2$

Example:

$f(x)=(x-1)(2-x)=-x^2+3x-2 \quad [a,b]=[1,\,2]$

Solution:

$x_0=a=1$

$x_n=b=2$

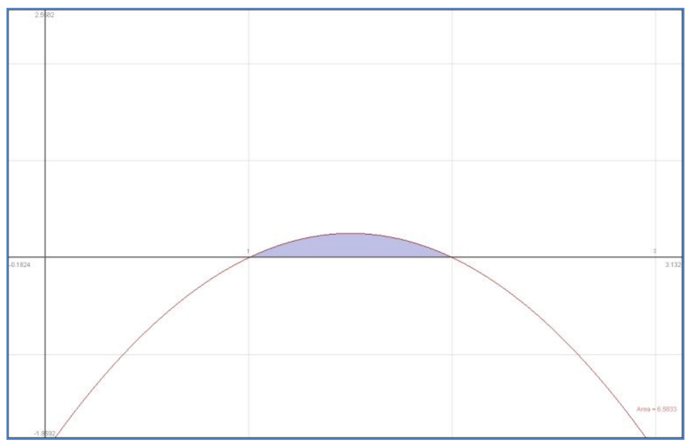
$\Delta x=\frac{b-a}{n}=\frac{2-1}{n}=\frac{1}{n}$

$x_i=x_0+i\Delta x=1+\frac{i}{n}$

$f(x_i)=-\left(1+\frac{i}{n}\right)^2+3\left(1+\frac{i}{n}\right)-2=-1-\frac{2i}{n}-\frac{i^2}{n^2}+3+\frac{3i}{n}-2=\frac{i}{n}-\frac{i^2}{n^2}$

Since $f(x)\geq 0$ when $x\in[1,\,2]$, the area will be equal to the limit of the sums of rectangles as the number of rectangles (n) goes to ∞.

$$
\begin{aligned}
A &= \lim_{n\to\infty} R_n \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}-\frac{i^2}{n^2}\right)\left(\frac{1}{n}\right) \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(i-\frac{i^2}{n}\right)\left(\frac{1}{n^2}\right) \\
&= \lim_{n\to\infty}\left(\frac{1}{n^2}\right)\sum_{i=1}^{n}\left(i-\frac{i^2}{n}\right) \\
&= \lim_{n\to\infty}\left(\frac{1}{n^2}\right)\left[\sum_{i=1}^{n} i-\sum_{i=1}^{n}\left(\frac{i^2}{n}\right)\right] \\
&= \lim_{n\to\infty}\left(\frac{1}{n^2}\right)\left[\sum_{i=1}^{n} i-\frac{1}{n}\sum_{i=1}^{n} i^2\right] \\
&= \lim_{n\to\infty}\left(\frac{1}{n^2}\right)\left[\frac{n(n+1)}{2}-\frac{1}{n}\left(\frac{n(n+1)(2n+1)}{6}\right)\right] \\
&= \lim_{n\to\infty}\left(\frac{1}{n^2}\right)\left[\frac{n(n+1)}{2}-\left(\frac{(n+1)(2n+1)}{6}\right)\right] \\
&= \lim_{n\to\infty}\left[\frac{n(n+1)}{2n^2}-\left(\frac{(n+1)(2n+1)}{6n^2}\right)\right] \\
&= \lim_{n\to\infty}\left[\frac{3n(n+1)}{6n^2}-\left(\frac{(n+1)(2n+1)}{6n^2}\right)\right] \\
&= \lim_{n\to\infty}\left[\frac{3n^2+3n}{6n^2}-\frac{2n^2+3n+1}{6n^2}\right] \\
&= \lim_{n\to\infty}\left[\frac{3n^2+3n-(2n^2+3n+1)}{6n^2}\right] \\
&= \lim_{n\to\infty}\left[\frac{n^2-1}{6n^2}\right] \\
&= \frac{1}{6}
\end{aligned}
$$



AREA = $\frac{1}{6}$ **square units**

Practice Problems

Using the limit definition of the area that the graph of a positive function forms with the horizontal axis and two vertical lines $x = a$ and $x = b$, find the exact area in each case:

1. $f(x) = (1-x)(x-2)(3-x) \quad [1,2]$
2. $f(x) = x - x^2 \quad [0,1]$
3. $f(x) = x^2 + x \quad [1, 4]$
4. $f(x) = -x^3 + 2x^2 \quad [0, 2]$
5. $y = 10 - x^2 \quad [1, 3]$