Interpretations of the term ‘AVERAGE’ in Calculus

This handout is aimed at summarizing some of the different contexts of the term ‘average’ as it is encountered in the AP Calculus curriculum.

Average Rate of Change

The average rate of change of a function $f(x)$ is always defined on a closed interval $[a,b]$, and is given by the quotient:

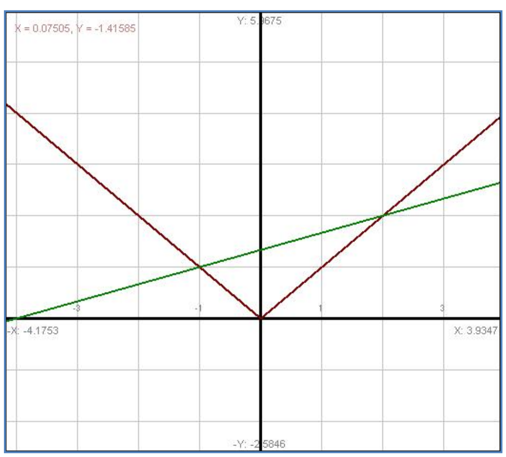
$$\frac{rise}{run} = \frac{f(b)-f(a)}{b-a} = \frac{f(a)-f(b)}{a-b} = \frac{y_2-y_1}{x_2-x_1}$$

Geometrically, this number represents the slope of the secant (segment) line connecting the points $(a, f(a))$ and $(b, f(b))$.

Example: The average rate of change of $f(x)=|x|$ on the interval $[-1,2]$ is given by

$$AVG_{RATE} = \frac{f(2)-f(-1)}{2-(-1)} = \frac{2-1}{3} = \frac{1}{3}$$

The slope of the secant line (green) connecting the two points is $\frac{1}{3}$. On average, a three unit increase in $x-$values led to a one unit increase in $y-$values.



Interpretation: The average rate of change of a function, as the term suggests, describes how fast the function is changing *on average* on a given interval. In the example we gave above, the function grew on average. Notice, however, that throughout the given interval the function actually exhibited both decreasing and increasing behavior. It decreased until it reached the origin, and increased thereafter.

Average Velocity of an Object

When the position function of an object is given by $s(t)$, we defined the average velocity over a time interval $[a,b]$ as the average rate of change of the position function:

$$V_{AVG:[a,b]} = \frac{s(b)-s(a)}{b-a} = \frac{s_2-s_1}{t_2-t_1}$$

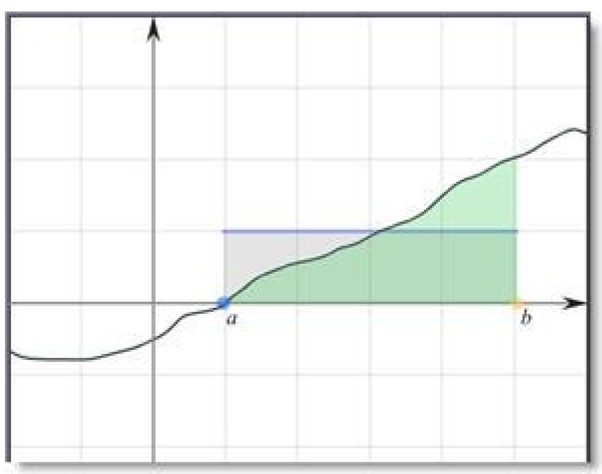
The interpretation is similar to that of the average rate of change, with appropriate units included ($\frac{m}{s}$, km/h, $\frac{mi}{h}$ etc.)

Average Value of a Continuous Function

The average value of a continuous function is always defined on an interval $[a,b]$, and is given by:

$$y_{AVG} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Geometrically, this value describes the typical $y-$value that the function takes on the given interval, given by the horizontal line in the sketch below:



The unit of the average value of a function is the same as the unit of the function ($y-$axis variable). So, if we measured the average value of a cost function $C(x)$ over the interval of production $[10,100]$, the unit of C_{AVG} would be $\$$.

Example: The average velocity of an object over a time interval $[a,b]$, when only the velocity function $v(t)$ is known, is given by:

$$v_{AVG} = \tfrac{1}{b-a}\int_a^b v(t)\,dt$$

Similarly, the average acceleration of an object over a time interval $[a,b]$, when only the acceleration function $a(t)$ is known, is given by:

$$a_{AVG} = \tfrac{1}{b-a}\int_a^b a(t)\,dt$$

By contrast, if we are asked to find the average acceleration over a time interval $[a,b]$ when only the velocity function $v(t)$ is given, we write:

$$a_{AVG} = \frac{v(b)-v(a)}{b-a}$$

Note that the Fundamental Theorem of Calculus states:

$$v(b)-v(a) = \int_a^v a(t)\,dt$$

If we divide both sides by the time elapsed $b-a$, we get both definitions of average acceleration over the interval $[a,b]$:

$$\frac{v(b)-v(a)}{b-a} = \tfrac{1}{b-a}\int_a^b a(t)\,dt$$

Instantaneous Rate of Change of a Function

The instantaneous rate of change of a function $f(x)$ is always defined at one point $x=a$ (hence "instantaneous"). It is also known as the value of the derivative (or slope), and when it exists, is given by:

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

This rate is a number whose unit is the ratio between the units of the y and x variables. The instantaneous rate of change of the cost function at $x = 100$ is written as $C'(100)$ and measured in dollars per units of output.