AP CALCULUS BC
Final Notes

Trigonometric Formulas

1. $\sin^2\theta+\cos^2\theta=1$
2. $1+\tan^2\theta=\sec^2\theta$
3. $1+\cot^2\theta=\csc^2\theta$
4. $\sin(-\theta)=-\sin\theta$
5. $\cos(-\theta)=\cos\theta$
6. $\tan(-\theta)=-\tan\theta$
7. $\sin(A+B)=\sin A\cos B+\sin B\cos A$
8. $\sin(A-B)=\sin A\cos B-\sin B\cos A$
9. $\cos(A+B)=\cos A\cos B-\sin A\sin B$
10. $\cos(A-B)=\cos A\cos B+\sin A\sin B$
11. $\sin 2\theta=2\sin\theta\cos\theta$
12. $\cos 2\theta=\cos^2\theta-\sin^2\theta=2\cos^2\theta-1=1-2\sin^2\theta$
13. $\tan\theta=\dfrac{\sin\theta}{\cos\theta}=\dfrac{1}{\cot\theta}$
14. $\cot\theta=\dfrac{\cos\theta}{\sin\theta}=\dfrac{1}{\tan\theta}$
15. $\sec\theta=\dfrac{1}{\cos\theta}$
16. $\csc\theta=\dfrac{1}{\sin\theta}$
17. $\cos(\frac{\pi}{2}-\theta)=\sin\theta$
18. $\sin(\frac{\pi}{2}-\theta)=\cos\theta$

Differentiation Formulas

1. $\dfrac{d}{dx}(x^n)=nx^{n-1}$
2. $\dfrac{d}{dx}(fg)=fg'+gf'$
3. $\dfrac{d}{dx}(\dfrac{f}{g})=\dfrac{gf'-fg'}{g^2}$
4. $\dfrac{d}{dx}f(g(x))=f'(g(x))g'(x)$
5. $\dfrac{d}{dx}(\sin x)=\cos x$
6. $\dfrac{d}{dx}(\cos x)=-\sin x$
7. $\dfrac{d}{dx}(\tan x)=\sec^2 x$
8. $\dfrac{d}{dx}(\cot x)=-\csc^2 x$
9. $\dfrac{d}{dx}(\sec x)=\sec x\tan x$
10. $\dfrac{d}{dx}(\csc x)=-\csc x\cot x$
11. $\dfrac{d}{dx}(e^x)=e^x$
12. $\dfrac{d}{dx}(a^x)=a^x\ln a$
13. $\dfrac{d}{dx}(\ln x)=\dfrac{1}{x}$
14. $\dfrac{d}{dx}(Arc\sin x)=\dfrac{1}{\sqrt{1-x^2}}$
15. $\dfrac{d}{dx}(Arc\tan x)=\dfrac{1}{1+x^2}$
16. $\dfrac{d}{dx}(Arc\sec x)=\dfrac{1}{|x|\sqrt{x^2-1}}$

17. $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ *Chain Rule*

Integration Formulas

1. $\int a\,dx = ax + C$
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$
3. $\int \frac{1}{x}\,dx = \ln|x| + C$
4. $\int e^x\,dx = e^x + C$
5. $\int a^x dx = \frac{a^x}{\ln a} + C$
6. $\int \ln x\,dx = x\ln x - x + C$
7. $\int \sin x\,dx = -\cos x + C$
8. $\int \cos x\,dx = \sin x + C$
9. $\int \tan x\,dx = \ln|\sec x| + C \text{ or } -\ln|\cos x| + C$
10. $\int \cot x\,dx = \ln|\sin x| + C$
11. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
12. $\int \csc x\,dx = \ln|\csc x - \cot x| + C = -\ln|\csc x + \cot x| + C$
13. $\int \sec^2 x\,dx = \tan x + C$
14. $\int \sec x \tan x\,dx = \sec x + C$
15. $\int \csc^2 x\,dx = -\cot x + C$
16. $\int \csc x \cot x\,dx = -\csc x + C$
17. $\int \tan^2 x\,dx = \tan x - x + C$
18. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} Arc\tan\left(\frac{x}{a}\right) + C$
19. $\int \frac{dx}{\sqrt{a^2 - x^2}} = Arc\sin\left(\frac{x}{a}\right) + C$
20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} Arc\sec\frac{|x|}{a} + C = \frac{1}{a} Arc\cos\left|\frac{a}{x}\right| + C$

Formulas and Theorems

1. <u>Limits and Continuity</u>:

A function $y = f(x)$ is <u>continuous</u> at $x = a$ if

i). *f(a)* exists

ii). $\lim\limits_{x \to a} f(x)$ exists

iii). $\lim\limits_{x \to a} = f(a)$

Otherwise, *f* is discontinuous at x = 1.

The limit $\lim\limits_{x \to a} f(x)$ exists if and only if both corresponding one-sided limits exist and are equal – that is,

$$\lim_{x \to a} f(x) = L \Leftrightarrow \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$$

2. <u>Even and Odd Functions</u>

1. A function $y = f(x)$ is <u>even</u> if $f(-x) = f(x)$ for every x in the function's domain. Every even function is symmetric about the y-axis.
2. A function $y = f(x)$ is <u>odd</u> if $f(-x) = -f(x)$ for every x in the function's domain. Every odd function is symmetric about the origin.

3. <u>Periodicity</u>

A function $f(x)$ is periodic with period $p\,(p > 0)$ if $f(x + p) = f(x)$ for every value of x.

<u>Note</u>: The period of the function $y = A\sin(Bx + C)$ or $y = A\cos(Bx + C)$ is $\dfrac{2\pi}{|B|}$.

The amplitude is $|A|$. The period of $y = \tan x$ is π.

4. <u>Intermediate-Value Theorem</u>

A function $y = f(x)$ that is continuous on a closed interval $[a,b]$ takes on every value between $f(a)$ and $f(b)$.

<u>Note</u>: If f is continuous on $[a,b]$ and $f(a)$ and $f(b)$ differ in sign, then the equation $f(x) = 0$ has at least one solution in the open interval (a,b).

5. <u>Limits of Rational Functions as $x \to \pm\infty$</u>

i). $\lim\limits_{x \to \pm\infty} \dfrac{f(x)}{g(x)} = 0$ if the degree of $f(x) <$ the degree of $g(x)$

<u>Example</u>: $\lim\limits_{x \to \infty} \dfrac{x^2 - 2x}{x^3 + 3} = 0$

ii). $\lim\limits_{x \to \pm\infty} \dfrac{f(x)}{g(x)}$ is infinite if the degrees of $f(x) >$ the degree of $g(x)$

<u>Example</u>: $\lim\limits_{x \to \infty} \dfrac{x^3 + 2x}{x^2 - 8} = \infty$

iii). $\lim\limits_{x \to \pm\infty} \dfrac{f(x)}{g(x)}$ is finite if the degree of $f(x) =$ the degree of $g(x)$

Example: $\lim\limits_{x \to \infty} \dfrac{2x^2 - 3x + 2}{10x - 5x^2} = -\dfrac{2}{5}$

Horizontal and Vertical Asymptotes

1. A line $y = b$ is a horizontal asymptote of the graph $y = f(x)$ if either $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$.

2. A line $x = a$ is a vertical asymptote of the graph $y = f(x)$ if either $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} = \pm\infty$.

6. Average and Instantaneous Rate of Change

i). *Average Rate of Change*: If (x_0, y_0) and (x_1, y_1) are points on the graph of $y = f(x)$, then the average rate of change of y with respect to x over the interval $[x_0, x_1]$ is $\dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = \dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{\Delta y}{\Delta x}$.

ii). *Instantaneous Rate of Change*: If (x_0, y_0) is a point on the graph of $y = f(x)$, then the instantaneous rate of change of y with respect to x at x_0 is $f'(x_0)$.

7. Definition of Derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{of} \quad f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The latter definition of the derivative is the instantaneous rate of change of $f(x)$ with respect to x at x = a.
Geometrically, the derivative of a function at a point is the slope of the tangent line to the graph of the function at that point.

8. The Number ***e*** as a limit

i). $$\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = e$$

ii). $$\lim_{n \to 0} \left(1 + \frac{n}{1}\right)^{\frac{1}{n}} = e$$

9. Rolle's Theorem

If f is continuous on $[a,b]$ and differentiable on (a,b) such that $f(a) = f(b)$, then there is at least one number c in the open interval (a,b) such that $f'(c) = 0$.

10. Mean Value Theorem

If f is continuous on $[a,b]$ and differentiable on (a,b), then there is at least one number c in (a,b) such that $\dfrac{f(b) - f(a)}{b - a} = f'(c)$.

11. Extreme-Value Theorem

If f is continuous on a closed interval $[a,b]$, then $f(x)$ has both a maximum and minimum on $[a,b]$.

12. To find the maximum and minimum values of a function $y = f(x)$, locate

1. the points where $f'(x)$ is zero or where $f'(x)$ fails to exist.
2. the end points, if any, on the domain of $f(x)$.

Note: These are the only candidates for the value of x where $f(x)$ may have a maximum or a
minimum.

13. Let f be differentiable for $a < x < b$ and continuous for a $a \le x \le b$,
   1. If $f'(x) > 0$ for every x in (a,b), then f is increasing on $[a,b]$.
   2. If $f'(x) < 0$ for every x in (a,b), then f is decreasing on $[a,b]$.

15. Suppose that $f''(x)$ exists on the interval (a,b)
   1. If $f''(x) > 0$ in (a,b), then f is concave upward in (a,b).
   2. If $f''(x) < 0$ in (a,b), then f is concave downward in (a,b).

   To locate the points of inflection of $y = f(x)$, find the points where $f''(x) = 0$ <u>or</u> where $f''(x)$ fails to exist. These are the only candidates where $f(x)$ may have a point of inflection. Then test these points to make sure that $f''(x) < 0$ on one side and $f''(x) > 0$ on the other.

16a. If a function is differentiable at point $x = a$, it is continuous at that point. The converse is false,
in other words, continuity does <u>not</u> imply differentiability.

16b. <u>Local Linearity and Linear Approximations</u>
The linear approximation to $f(x)$ near $x = x_0$ is given by $y = f(x_0) + f'(x_0)(x - x_0)$ for x sufficiently close to x_0.

To estimate the slope of a graph at a point – just draw a tangent line to the graph at that point. Another way is (by using a graphing calculator) to "zoom in" around the point in question until the graph "looks" straight. This method almost always works. If we "zoom in" and the graph looks straight at a point, say $(a, f(a))$, then the function is <u>locally linear</u> at that point.

The graph of $y = |x|$ has a sharp corner at x = 0. This corner cannot be smoothed out by "zooming in" repeatedly. Consequently, the derivative of $|x|$ does not exist at x = 0, hence, is <u>not</u> locally linear at x = 0.

17. <u>Dominance and Comparison of Rates of Change</u>
Logarithm functions grow slower than any power function (x^n).

Among power functions, those with higher powers grow faster than those with lower powers.

All power functions grow slower than any exponential function $(a^x, a > 1)$.

Among exponential functions, those with larger bases grow faster than those with smaller bases.

We say, that as $x \to \infty$:

1. $f(x)$ grows faster than $g(x)$ if $\lim_{x\to\infty} \frac{f(x)}{g(\ )} = \infty$ or if $\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0$.

   If $f(x)$ grows faster than $g(x)$ as $x \to \infty$, then $g(x)$ grows slower than $f(x)$ as $x \to \infty$.

2. $f(x)$ and $g(x)$ grow at the same rate as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \neq 0$ (L is finite and nonzero).

For example,

1. e^x grows faster than x^3 as $x \to \infty$ since $\lim_{x\to\infty} \frac{e^x}{x^3} = \infty$
2. x^4 grows faster than $\ln x$ as $x \to \infty$ since $\lim_{x\to\infty} \frac{x^4}{\ln x} = \infty$
3. $x^2 + 2x$ grows at the same rate as x^2 as $x \to \infty$ since $\lim_{x\to\infty} \frac{x^2 + 2x}{x^2} = 1$

To find some of these limits as $x \to \infty$, you may use the graphing calculator. Make sure that an appropriate viewing window is used.

18. L'Hôpital's Rule

If $\lim_{x \to a} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and if $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

19. Inverse function

1. If f and g are two functions such that $f(g(x)) = x$ for every x in the domain of g and $g(f(x)) = x$ for every x in the domain of f, then f and g are inverse functions of each other.
2. A function f has an inverse if and only if no horizontal line intersects its graph more than once.
3. If f is either increasing or decreasing in an interval, then f has an inverse.
4. If f is differentiable at every point on an interval I, and $f'(x) \neq 0$ on I, then $g = f^{-1}(x)$ is differentiable at every point of the interior of the interval $f(I)$ and $g'(f(x)) = \frac{1}{f'(x)}$.

20. Properties of $y = e^x$

1. The exponential function $y = e^x$ is the inverse function of $y = \ln x$.
2. The domain is the set of all real numbers, $-\infty < x < \infty$.
3. The range is the set of all positive numbers, $y > 0$.
4. $\frac{d}{dx}(e^x) = e^x$
5. $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$
6. $y = e^x$ is continuous, increasing, and concave up for all x.

7. $\lim_{x \to +\infty} e^x = +\infty$ and $\lim_{x \to -\infty} e^x = 0$.

8. $e^{\ln x} = x$, for $x > 0$; $\ln(e^x) = x$ for all x.

21. Properties of $y = \ln x$

1. The domain of $y = \ln x$ is the set of all positive numbers, $x > 0$.
2. The range of $y = \ln x$ is the set of all real numbers, $-\infty < y < \infty$.
3. $y = \ln x$ is continuous and increasing everywhere on its domain.
4. $\ln(ab) = \ln a + \ln b$.
5. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$.
6. $\ln a^r = r \ln a$.
7. $y = \ln x < 0$ if $0 < x < 1$.
8. $\lim_{x \to +\infty} \ln x = +\infty$ and $\lim_{x \to 0^+} \ln x = -\infty$.
9. $\log_a x = \frac{\ln x}{\ln a}$

22. Trapezoidal Rule

If a function f is continuous on the closed interval $[a,b]$ where $[a,b]$ has been partitioned into n subintervals $[x_0, x_1], [x_1, x_2], \ldots [x_{n-1}, x_n]$, each length $\frac{b-a}{n}$, then

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right].$$

23a. <u>Definition of Definite Integral as the Limit of a Sum</u>

Suppose that a function $f(x)$ is continuous on the closed interval $[a,b]$. Divide the interval into n equal subintervals, of length $\Delta x = \frac{b-a}{n}$. Choose one number in each subinterval, in other words, x_1 in the first, x_2 in the second, …, x_k in the k th,…, and x_n in the n th. Then $\lim_{n \to \infty} \sum_{k=1}^{n} f(x_k)\Delta x = \int_a^b f(x)\,dx = F(b) - F(a)$.

23b. <u>Properties of the Definite Integral</u>

Let $f(x)$ and $g(x)$ be continuous on $[a,b]$.

i). $\int_a^b c \cdot f(x)\,dx = c\int_a^b f(x)\,dx$ for any constant c.

ii). $\int_a^a f(x)\,dx = 0$

iii). $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

iv). $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where f is continuous on an interval containing the numbers a, b, and c.

v). If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)\,dx = 0$

vi). If $f(x)$ is an even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

vii). If $f(x) \ge 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \ge 0$

viii). If $g(x) \ge f(x)$ on $[a,b]$, then $\int_a^b g(x)\,dx \ge \int_a^b f(x)\,dx$

24. <u>Fundamental Theorem of Calculus:</u>

$\int_a^b f(x)\,dx = F(b) - F(a)$, where $F'(x) = f(x)$, or $\frac{d}{dx}\int_a^b f(x)\,dx = f(x)$.

25. <u>Second Fundamental Theorem of Calculus</u>:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \quad \text{or} \quad \frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(x) \cdot g'(x)$$

26. Velocity, Speed, and Acceleration

1. The velocity of an object tells how fast it is going and in which direction. Velocity is an instantaneous rate of change.
2. The speed of an object is the absolute value of the velocity, $|v(t)|$. It tells how fast it is going disregarding its direction.
   The speed of a particle increases (speeds up) when the velocity and acceleration have the same signs. The speed decreases (slows down) when the velocity and acceleration have opposite signs.
3. The acceleration is the instantaneous rate of change of velocity – it is the derivative of the velocity – that is, $a(t) = v'(t)$. Negative acceleration (deceleration) means that the velocity is decreasing. The acceleration gives the rate ot which the velocity is changing.

Therefore, if x is the displacement of a moving object and t is time, then:

i) velocity = $v(t) = x'(t) = \frac{dx}{dt}$

ii) acceleration = $a(t) = x''(t) = v'(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

iii) $v(t) = \int a(t)\,dt$

iv) $x(t) = \int v(t)\,dt$

Note: The average velocity of a particle over the time interval from t_0 to another time t, is Average Velocity = $\frac{\text{Change in position}}{\text{Length of time}} = \frac{s(t) - s(t_0)}{t - t_0}$, where $s(t)$ is the position of the particle at time t.

27. The average value of $f(x)$ on $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

28. Area Between Curves

If f and g are continuous functions such that $f(x) \geq g(x)$ on $[a,b]$, then area between the curves is $\int_a^b [f(x) - g(x)]\,dx$.

29. Integration By "Parts"

If $u = f(x)$ and $v = g(x)$ and if $f'(x)$ and $g'(x)$ are continuous, then $\int u\,dx = uv - \int v\,du$.

Note: The goal of the procedure is to choose u and dv so that $\int v\,du$ is easier to solve
than the original problem.

Suggestion:

When "choosing" u, remember **L.I.A.T.E**, where **L** is the logarithmic function, **I** is an inverse trigonometric function, **A** is an algebraic function, **T** is a trigonometric function, and **E** is the exponential function. Just choose u as the first expression in **L.I.A.T.E** (and dv will be the remaining part of the integrand). For example, when integrating $\int x \ln x \, dx$, choose $u = \ln x$ since **L** comes first in **L.I.A.T.E**, and $dv = x \, dx$. When integrating $\int xe^x \, dx$, choose $u = x$, since x is an algebraic function, and **A** comes before **E** in **L.I.A.T.E**, and $dv = e^x dx$. One more example, when integrating $\int x \, Arc\tan(x) \, dx$, let $u = Arc\tan(x)$, since **I** comes before **A** in **L.I.A.T.E**, and $dv = x \, dx$.

30. Volume of Solids of Revolution (rectangles drawn perpendicular to the axis of revolution)

Let f be nonnegative and continuous on $[a,b]$, and let R be the region bounded above by $y = f(x)$, below by the x-axis and the sides by the lines $x = a$ and $x = b$.

1. When this region R is revolved about the x-axis, it generates a solid (having circular cross sections) whose volume $V = \pi\int_a^b [f(x)]^2\, dx$.
2. When two functions are involved: $V = \pi\int_a^b \left(r_o^2 - r_i^2\right) dx$ where r_o is the distance between the axis of revolution and the furthest side of the shaded region and r_i is the distance between the axis of revolution and the nearest side of the shaded region.
3. When the rectangles are perpendicular to the x-axis, the integral will be in terms of x.
   When the rectangles are perpendicular to the y-axis, the integral will be in terms of y.

30b. Volume of Solids with Known Cross Sections

1. For cross sections of area $A(x)$, taken perpendicular to the x-axis, volume = $\int_a^b A(x)\, dx$.

   Volumes on the interval [a, b] where $a(x)$ is the length of a side of the section:

   Square: $V = \int_a^b [a(x)]^2\, dx$ Equilateral Triangle: $V = \frac{\sqrt{3}}{4}\int_a^b [a(x)]^2\, dx$

   Semi-circle: $V = \frac{\pi}{8}\int_a^b [a(x)]^2\, dx$

   Isosceles Right Triangle: $V = \frac{1}{2}\int_a^b [a(x)]^2\, dx$ (when a = leg of triangle)

   Isosceles Right Triangle: $V = \frac{1}{4}\int_a^b [a(x)]^2\, dx$ (when a = hypotenuse of triangle)
2. For cross sections of area $A(y)$, taken perpendicular to the y-axis, volume = $\int_a^b A(y)\, dy$.

30c. Shell Method (rectangles drawn parallel to the axis of revolution)

1. Horizontal Axis of Revolution: $V = 2\pi\int_c^d p(y)h(y)\,dy$ (p is the distance between the axis of revolution and the center of a rectangle.)

2. Vertical Axis of Revolution: $V = 2\pi\int_a^b p(x)h(x)dx$ (p is the distance between the axis of revolution and the center of a rectangle.)

31. Solving Differential Equations: Graphically and Numerically

Slope Fields

At every point (x, y) a differential equation of the form $\frac{dy}{dx} = f(x, y)$ gives the slope of the member of the family of solutions that contains that point. A slope field is a graphical representation of this family of curves. At each point in the plane, a short segment is drawn whose slope is equal to the value of the derivative at that point. These segments are tangent to the solution's graph at the point.

The slope field allows you to sketch the graph of the solution curve even though you do not have its equation. This is done by starting at any point (usually the point given by the initial condition), and moving from one point to the next in the direction indicated by the segments of the slope field.

Some calculators have built in operations for drawing slope fields; for calculators without this feature there are programs available for drawing them.

Euler's Method

Euler's Method is a way of approximating points on the solution of a differential equation $\frac{dy}{dx} = f(x, y)$. The calculation uses the tangent line approximation to move from one point to the next. That is, starting with the given point (x_1, y_1) – the initial condition, the point $(x_1 + \Delta x, y_1 + f'(x_1, y_1)\Delta x)$ approximates a nearby point on the solution graph. This aproximation may then be used as the starting point to calculate a third point and so on. The accuracy of the method decreases with large values of Δx. The error increases as each successive point is used to find the next. Calculator programs are available for doing this calculation.

32. Logistics

1. Rate is jointly proportional to its size and the difference between a fixed positive number (L) and its size.

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right) \text{ which yields}$$

$$y = \frac{L}{1 + Ce^{-kt}} \text{ through separation of variables}$$

2. $\lim_{t\to\infty} y = L$; L = carrying capacity (Maximum); horizontal asymptote

3. y-coordinate of inflection point is $\frac{L}{2}$, i.e. when it is growing the fastest (or max rate).

33. Definition of Arc Length

If the function given by $y = f(x)$ represents a smooth curve on the interval $[a,b]$, then the arc length of f between a and b is given by $s = \int_a^b \sqrt{1+[f'(x)]^2}\, dx$.

34. Improper Integral

$\int_a^b f(x)\, dx$ is an improper integral if

1. f becomes infinite at one or more points of the interval of integration, or
2. one or both of the limits of integration is infinite, or
3. both (1) and (2) hold.

35. Parametric Form of the Derivative

If a smooth curve **C** is given by the parametric equations $x = f(x)$ and $y = g(t)$, then the slope of the curve **C** at (x, y) is $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$, $\frac{dx}{dt} \neq 0$.

Note: The second derivative, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dt}\left[\frac{dy}{dx}\right] \div \frac{dx}{dt}$.

36. Arc Length in Parametric Form

If a smooth curve **C** is given by $x = f(t)$ and $y = g(t)$ and these functions have continuous first derivatives with respect to t for $a \le t \le b$, and if the point $P(x, y)$ traces the curve exactly once as t moves from $t = a$ to $t = b$, then the length of the curve is given by $s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$.

$\text{speed} = \sqrt{[f'(t)]^2 + [g'(t)]^2}$

37. Polar Coordinates

1. Cartesian vs. Polar Coordinates. The polar coordinates (r,θ) are related to the Cartesian coordinates (x,y) as follows:

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

$$\tan\theta = \frac{y}{x} \text{ and } x^2 + y^2 = r^2$$

2. To find the points of intersection of two polar curves, find (r,θ) satisfying the first equation for which some points $(r,\theta+2n\pi)$ or $(-r,\theta+\pi+2n\pi)$ satisfy the second equation. Check separately to see if the origin lies on both curves, i.e. if r can be 0. Sketch the curves.
3. Area in Polar Coordinates: If f is continuous and nonnegative on the interval $[\alpha,\beta]$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by

$$A = \frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2\,d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta$$

4. Derivative of Polar function: Given $r = f(\theta)$, to find the derivative, use parametric equations.

$$x = r\cos\theta = f(\theta)\cos\theta \quad \text{and} \quad y = r\sin\theta = f(\theta)\sin\theta.$$

Then $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$$

5. Arc Length in Polar Form: $$s = \int_{\alpha}^{\beta}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

38. Sequences and Series

1. If a sequence $\{a_n\}$ has a limit L, that is, $\lim_{n\to\infty} a_n = L$, then the sequence is said to <u>converge</u> to L. If there is no limit, the series <u>diverges</u>. If the sequence $\{a_n\}$ converges, then its limit is unique. Keep in mind that

   $\lim_{n\to\infty} \frac{\ln n}{n} = 0$; $\lim_{n\to\infty} x^{\left(\frac{1}{n}\right)} = 1$; $\lim_{n\to\infty} \sqrt[n]{n} = 1$; $\lim_{n\to\infty} \frac{x^n}{n!} = 0$. These limits are useful and arise frequently.

2. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges; the geometric series $\sum_{n=0}^{\infty} ar^n$ converges to $\frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$ and $a \neq 0$.

3. The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

4. <u>Limit Comparison Test</u>: Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be a series of nonnegative terms, with $a_n \neq 0$ for all sufficiently large n, and suppose that $\lim_{n\to\infty} \frac{b_n}{a_n} = c > 0$. Then the two series either both converge or both diverge.

5. <u>Alternating Series</u>: Let $\sum_{n=1}^{\infty} a_n$ be a series such that

   i) the series is alternating
   ii) $|a_{n+1}| \leq |a_n|$ for all n, and
   iii) $\lim_{n\to\infty} a_n = 0$

   Then the series *converges*.

   <u>Alternating Series Remainder</u>: The remainder R_N is less than (or equal to) the first neglected term

$$|R_N| \leq a_{N+1}$$

6. <u>The *n*-th Term Test for Divergence</u>: If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges. Note that the converse is *false*, that is, if $\lim_{n\to\infty} a_n = 0$, the series may or may not converge.

7. A series $\sum a_n$ is <u>absolutely convergent</u> if the series $\sum |a_n|$ converges. If $\sum a_n$ converges, but $\sum |a_n|$ does not converge, then the series is <u>conditionally convergent</u>. Keep in mind that if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

8. <u>Comparison Test</u>: If $0 \le a_n \le b_n$ for all sufficiently large n, and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

9. <u>Integral Test</u>: If $f(x)$ is a positive, continuous, and decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ will converge if the improper integral $\int_1^{\infty} f(x)\,dx$ converges. If the improper integral $\int_1^{\infty} f(x)\,dx$ diverges, then the infinite series $\sum_{n=1}^{\infty} a_n$ diverges.

10. <u>Ratio Test</u>: Let $\sum a_n$ be a series with nonzero terms.

    i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then the series converges absolutely.

    ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, then the series is divergent.

    iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then the test is inconclusive (and another test must be used).

11. <u>Power Series</u>: A power series is a series of the form
$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots$ or
$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n + \ldots$ in which the center a and the coefficients $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants. The set of all numbers x for which the power series converges is called the <u>interval of convergence</u>.

12. <u>Taylor Series</u>: Let f be a function with derivatives of all orders throughout some intervale containing a as an interior point. Then the Taylor series generated by f

at a is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

The remaining terms after the term containing the nth derivative can be expressed as a remainder to Taylor's Theorem:

$$f(x) = f(a) + \sum_{1}^{n} f^{(n)}(a)(x-a)^n + R_n(x) \text{ where } R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt$$

Lagrange's form of the remainder: $R_n x = \dfrac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$, where $a < c < x$.

The series will converge for all values of x for which the remainder approaches zero as $x \to \infty$.

13. Frequently Used Series and their Interval of Convergence

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)x^{2n}}{(2n)!}, \quad |x| < \infty$$