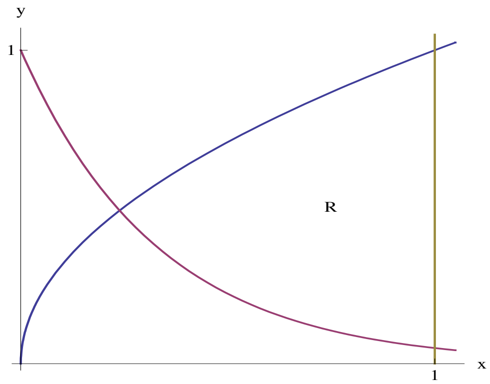
BC Practice Exam: Free Response, Part I. Graphing calculators may be used.

---

1. Let R be the region bounded by the graphs of $y = \sqrt{x}$ and $y = e^{-3x}$, and the vertical line $x = 1$. See the figure.



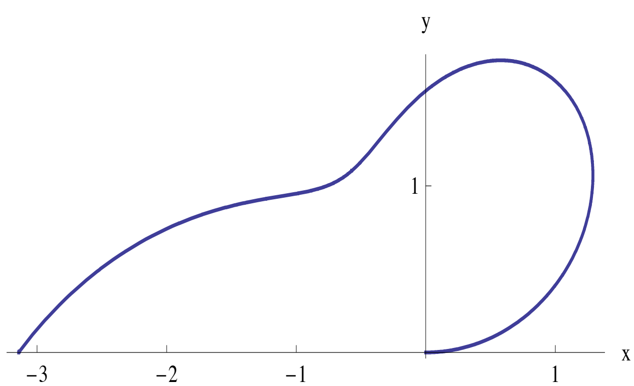
   (a) Find the area of R.

   (b) Find the volume of the solid generated when R is revolved about the horizontal line $y = 1$.

   (c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a rectangle whose height is 5 times the length of its base in R. Find the volume of this solid.

---

2. The curve shown below is the graph of the polar equation $r = \theta + \sin 2\theta$, for $0 \le \theta \le \pi$.



   (a) Find the area bounded by the curve and the x-axis.

   (b) Find the angle θ that corresponds to the point on the curve with x-coordinate -2.

   (c) For $\dfrac{\pi}{3} < \theta < \dfrac{2\pi}{3}$, $\dfrac{dr}{d\theta}$ is negative. What does this say about r? What does this say about the curve?

   (d) Find the value of θ in the interval $0 \le \theta \le \dfrac{\pi}{2}$ that corresponds to the point on the curve with greatest distance from the origin. Justify your answer.

---

3. The velocity of an object in motion in the plane for $0 \le t \le 1$ is given by the vector

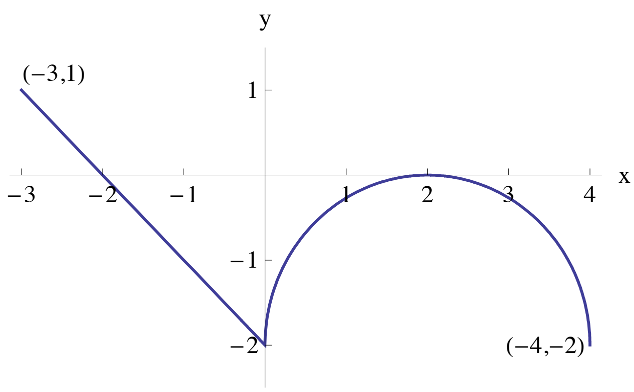
$$\mathbf{v}(t) = \left(\frac{1}{\sqrt{4-t^2}}, \frac{t}{\sqrt{4-t^2}} \right).$$

(a) When is the object at rest?

(b) If this object was at the origin when $t = 0$, what are its speed and position when $t = 1$?

(c) Find an equation of the curve the object follows, expressing y as a function of x.

4. Let f be a function defined on the closed interval $-3 \le x \le 4$ with $f(0) = 3$. The graph of f', the derivative of f, consists of a line segment and a semi-circle, as shown in the figure.



(a) On what intervals, if any, is f increasing? Justify your answer.

(b) Find the x-coordinate of each point of inflection of the graph of f on the open interval $-3 < x < 4$. Justify your answer.

(c) Find an equation for the line tangent to the graph of f at the point $(0, 3)$.

(d) Find $f(-3)$ and $f(4)$. Show the work that leads to your answer.

---

5. Consider the differential equation

$$\frac{dy}{dx} = 5x^2 - \frac{6}{y-2}, \qquad y \neq 2.$$

Let $y = f(x)$ be the particular solution of this differential equation which satisfies the initial condition $f(-1) = -4$.

(a) Evaluate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $(-1, -4)$.

(b) Is it possible for the x-axis to be tangent to the graph of f at some point? Explain why or why not.

(c) Find the second degree Taylor polynomial for f about $x = -1$.

(d) Use Euler's method, starting at $x = -1$ with two steps of equal size, to approximate $f(0)$. Show the work that leads to your answer.

---

6. The function f is defined by the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

for all real numbers x.

(a) Find $f'(0)$ and $f''(0)$. Determine whether f has a local maximum, a local minimum, or neither at $x = 0$. Give a reason for your answer.

(b) Show that $1 - \frac{1}{3!}$ approximates $f(1)$ with error less than $\frac{1}{100}$.

(c) Show that $y = f(x)$ is a solution to the differential equation $xy' + y = \cos x$.

Problem 1

a)

Intersection point: $x \approx 0.238734 = m$

To find the area we integrate

$$\int_{0.238734}^{1} \left(\sqrt{x} - e^{-3x}\right) dx \approx 0.44263 = 0.443$$

b)

We use the Washer Method, since there is a gap between the region and the rotational axis.

$$\int_{m}^{1} \pi\left[\left(1-e^{-3x}\right)^2 - \left(1-\sqrt{x}\right)^2\right] dx \approx 1.42356 = 1.424$$

c)

$$\int_{m}^{1} A(x)\,dx = \int_{m}^{1} \left[\left(\sqrt{x} - e^{-3x}\right) * 5\left(\sqrt{x} - e^{-3x}\right)\right] dx \approx 1.55435 = 1.554$$

Problem 2

a)

$$\int_{\theta=0}^{\theta=\pi} \frac{1}{2} r^2\, d\theta = \int_{\theta=0}^{\theta=\pi} \frac{1}{2}\left(\theta + \sin 2\theta\right)^2 d\theta = 4.3821$$

b)

$$x = -2$$
$$x = r\cos\theta = (\theta + \sin 2\theta)\cos\theta$$
$$(\theta + \sin 2\theta)\cos\theta = -2$$
$$TI-84 : \theta = 2.78606$$

Note that this result agrees with the graph of the polar curve (second quadrant).

iLearnMath.net

c)

This suggests that r is getting smaller. The curve is getting closer to the pole, as confirmed by the graph.

d)

$Max(r(\theta))$

$r'(\theta) = 1 + 2\cos 2\theta \rightarrow \cos 2\theta = \frac{-1}{2} \rightarrow 2\theta = \frac{2\pi}{3} \rightarrow \theta = \frac{\pi}{3}$

$r''(\theta) = -4\sin 2\theta \rightarrow r''(\pi/3) < 0$

By the second derivative test, we have a local maximum. Use the Closed Interval Method to justify that this is the absolute maximum on the given closed interval. Evaluate at the endpoints and compare with the distance at the critical number found above. Visually, we confirm that the critical number does in fact give the largest distance from the pole.

Problem 3

a)

The object is at rest when the horizontal and vertical velocities are simultaneously equal to zero. Note that the horizontal velocity is never equal to zero, therefore the object in this model is never at rest.

b)

$\left(x(0), y(0)\right) = (0,0)$

$speed = \sqrt{\left(x'(1)\right)^2 + \left(y'(1)\right)^2} = \sqrt{\frac{1}{3} + \frac{1}{3}} = \sqrt{\frac{2}{3}}$

$x(1) - x(0) = \int_0^1 x'(t)\,dt = \int_0^1 \frac{1}{\sqrt{4-t^2}}\,dt \rightarrow x(1) = \int_0^1 \frac{1}{\sqrt{4-t^2}}\,dt = \frac{\pi}{6} = 0.523599$

$y(1) - y(0) = \int_0^1 y'(t)\,dt = \int_0^1 \frac{t}{\sqrt{4-t^2}}\,dt \rightarrow y(1) = \int_0^1 \frac{t}{\sqrt{4-t^2}}\,dt = 2 - \sqrt{3} = 0.267949$

$\left(x(1), y(1)\right) = \left(\frac{\pi}{6}, 2 - \sqrt{3}\right)$

iLearnMath.net

c)
To find a Cartesian equation, we need to eliminate the parameter t.

$$x(t) = x(0) + \int_0^t x'(w)\,dw = \int_0^t \frac{1}{\sqrt{4-w^2}}\,dw = \arcsin\left(\frac{t}{2}\right)$$

$$y(t) = y(0) + \int_0^t y'(w)\,dw = \int_0^t \frac{w}{\sqrt{4-w^2}}\,dw = -\sqrt{4-t^2}$$

$$t = 2\sin x \to y = -\sqrt{4-4\sin^2 x} = -2|\cos x|$$

$$y = -2|\cos x|$$

Problem 4

a)
f is increasing whenever its derivative is positive. This occurs on the interval $(-3,-2)$.
b)
f has a point of inflection whenever its second derivative changes sign while the first derivative maintains its sign. The third condition is that the original function needs to be continuous. This is satisfied because the function is differential on the given domain.
The second derivative (slope of the slope function) changes at $x = 0, x = 2$. These are the $x-$coordinates of the inflection points. Note the first derivative does not change sign at this two points. (It remains negative.)
c)

$$f(0) = 3$$

$$f'(0) = -2$$

$$y - 3 = -2(x-0) \to y = -2x + 3$$

d)

$$f(-3) - f(0) = \int_0^{-3} f'(x)\,dx$$

$$f(-3) = f(0) + \int_0^{-3} f'(x)\,dx = f(0) - \int_{-3}^{0} f'(x)\,dx = 3 - (0.5 - 2) = 4.5$$

$$f(4) - f(0) = \int_0^4 f'(x)\,dx$$

$$f(4) = f(0) + \int_0^4 f'(x)\,dx = 3 + [-8 + \frac{\pi r^2}{2}] = 3 - 8 + 2\pi = 2\pi - 5$$

Note: we used geometry to compute the integrals (two triangles, one rectangle, and one semi-circle)

iLearnMath.net

Problem 5

a)

$$\left.\frac{dy}{dx}\right|_{(-1,-4)} = 5 - \frac{6}{-4-2} = 5+1 = 6$$

$$\frac{d^2y}{dx^2} = 10x + \frac{6}{(y-2)^2}\frac{dy}{dx}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(-1,-4)} = -10 + \frac{6}{(-4-2)^2} * 6 = -9$$

b)

If the axis is a tangent to the graph, then we must satisfy:

$$\begin{cases} \frac{dy}{dx} = 0 \\ y = 0 \end{cases} \rightarrow 5x^2 - \frac{6}{0-2} = 0 \rightarrow 5x^2 = -3$$

Therefore, the described feature is impossible.

c)

$$T_2(x) = f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2!}(x+1)^2$$

$$T_2(x) = -4 + 6(x+1) + \frac{-9}{2}(x+1)^2$$

d)

x	y	dy/dx	Computations
-1	-4	6	$y-(-4) = 6(x+1) \rightarrow y(-0.5) = -1$
-0.5	-1	3.25	$y-(-1) = 3.25(x+0.5) \rightarrow y = 5/8 = 0.625$
0	0.625		

Using Euler's Method, $f(0)$ is approximately equal to 0.625.

iLearnMath.net

Problem 6

a)

$$f'(x) = \frac{-2x}{3!} + \frac{4x^3}{5!} - \frac{6x^5}{7!} + \ldots$$

$$\to f'(0) = 0$$

$$f''(x) = \frac{-2}{3!} + \frac{12x^2}{5!} - \frac{30x^3}{7!} + \ldots$$

$$\to f''(0) = \frac{-1}{3} < 0$$

By the second derivative test, the function $f(x)$ has a local maximum at $x = 0$.

b)

$$1 - \frac{1}{3!} = T_2(1)$$

Because the series is alternating and the terms absolutely converge to zero, the error in approximation is less than the absolute value of the next term:

$\frac{1}{5!} = \frac{1}{120} < \frac{1}{100}$, as wanted.

c)

The verification can be done in a couple of ways, one using the sigma notation, and the other using the first few terms. I discuss the second method here, and leave the sigma approach up to you to try.

$$xy' + y = \cos x$$

$$x[f'(x)] + y = x[\frac{-2x}{3!} + \frac{4x^3}{5!} - \frac{6x^5}{7!} + ..] + [1 - \frac{-x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + ..]$$

$$= [\frac{-2x^2}{3!} + \frac{4x^4}{5!} - \frac{6x^6}{7!} + ..] + [1 - \frac{-x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + ..] =$$

$$= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \ldots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \cos x,$$

as wanted.

Prepared by D. Shubleka | Last updated: 4/21/2013 11:27pm EST

iLearnMath.net