Problem 1

PART A.

$M'(7.5) \approxeq \frac{M(10)-M(5)}{10-5} = \frac{16-7}{5} = \frac{9}{5}$ birds per day per day.

PART B.

i.

$M_3 = 10 * M(5) + 10 * M(15) + 10 * M(25) = 10(7+6+2) = 150$

ii.

Approximately 150 male birds arrived at the nesting area over the 30-day interval. (Alternatively, the net change in the number of male birds present at the nesting area is an increase of 150 birds over the 30-day period.)

PART C.

$$\int_{15}^{45} F(t)dt = \int_{15}^{45} \left(18 + 16\sin(\tfrac{\pi}{20}(t+15))\right) dt = 540 + \frac{320}{\pi} \approxeq 641.859$$

Approximately 642 female birds arrive at the nesting area from $t = 15$ to $t = 45$.

PART D.

The functions $F(t)$ and $M(t)$ are differentiable and therefore continuous on the interval $[15, 20]$. The difference function $D(t)$ will therefore be continuous on the interval $[15, 20]$.

We apply the Intermediate Value Theorem:

$D(15) = M(15) - F(15) = 2 > 0$

$D(20) = M(20) - F(20) = 8\sqrt{2} - 13 \approxeq -1.686 < 0$

By the Intermediate Value Theorem, yes, $D(t) = 0$ at least once in interval $15 < t < 20$ since $D(t)$ switches sign in a continuous fashion during this interval.

Problem 2

PART A.

$r(\theta) = 3 + 2\sin(2\theta) + \cos(2\theta)$ on $[0, \pi]$.

$A(S) = \frac{1}{2} \int\limits_{0}^{\pi} (r(\theta))^2 d\theta \approxeq 18.064.$

PART B.

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-3}{7}$

Since we know that $\frac{dy}{d\theta} = \frac{3\sqrt{2}}{2}$, we can find:

$\frac{dx}{d\theta} = \frac{dy/d\theta}{dy/dx} = \frac{3\sqrt{2}}{2} * \frac{-7}{3} = \frac{-7\sqrt{2}}{2}.$

PART C.

r has a critical point when $r'(\theta)$ equals zero or is undefined. Using desmos or the graphing calculator, we find:

$\theta^* \approxeq 0.554$

Using the Second Derivative Test for Local Extrema, since $r''(\theta^*) \approxeq -8.944 < 0$ and $r'(\theta^*) = 0$, it follows that r has a relative maximum at $\theta = \theta^*$.

PART D.

The average distance from the origin is the average of value of $r(\theta)$. We set it up and compute with the graphing calculator:

average distance from the origin $= \frac{1}{\pi - \pi/2} \int\limits_{\pi/2}^{\pi} r(\theta) d\theta = \frac{2}{\pi} \int\limits_{\pi/2}^{\pi} r(\theta) d\theta \approxeq 1.727$

Problem 3

PART A.

The given slope field could not describe the differential equation

$$\frac{dH}{dt} = -\frac{1}{15}(H - 20)$$

because for $H > 20$ all the slopes should have a negative value and according to the graph, the line segments are positively-sloped.

PART B.

When $t = 0$, we know that the y-value is $H = 75$. The slope at this point can be measured by the differential equation:

$\frac{dH}{dt}|_{t=0,H=75} = \frac{-1}{15}(75 - 20) = \frac{-55}{15} = \frac{-11}{3}$

The tangent line has equation: $H - 75 = \frac{-11}{3}(t - 0) \rightarrow H(t) = 75 - \frac{11t}{3}$.

PART C.

The second derivative at $t = 0$ has a value of:

$\frac{d^2H}{dt^2}|_{t=0,H=75} = \frac{1}{225}(H - 20)|_{H=75} = \frac{55}{225} > 0$

This implies that the solution curve is concave up at the spot ($t = 0$) where the tangent line is drawn, therefore the approximation at the nearby point ($t = 5$) will be an underestimate.

PART D.

$\frac{1}{H-20}dH = \frac{-1}{15}dt \rightarrow \int \frac{1}{H-20}dH = \int \frac{-1}{15}dt$

$\ln|H - 20| = \frac{-1}{15}t + C$

Plug in $t = 0, H = 75$:

$\ln(55) = C$

$|H - 20| = e^{-t/15} * e^C = 55e^{-t/15} \rightarrow H - 20 = \pm 55e^{-t/15}$

Below we choose the positive option for the right side since $H > 20$.

$H(t) = 20 + 55e^{-t/15}$

Problem 4

PART A

$g(x) = f(x) - \ln x \to g'(x) = f'(x) - \frac{1}{x}$

$g'(2) = f'(2) - \frac{1}{2} = 1.5 - 0.5 = 1$

PART B

The graph of f has a point of inflection when 1) $f''(x)$, the derivative (slope function) of $f'(x)$ changes sign, 2) $f'(x)$ does not change sign, and 3) $f(x)$ is continuous.

All three conditions are met at $x = 1$.

PART C

f is increasing and concave down whenever $f'(x) > 0$ and $f''(x) < 0$. These conditions hold true on the interval $(1, 3)$

PART D

By FTC: $f(x) - f(2) = \int_2^x f'(t)dt \to f(x) = f(2) + \int_2^x f'(t)dt = 3 + \int_2^x f'(t)dt$

There are two critical numbers: $f'(x) = 0 \to x = -2$ or $x = 3$.

To find the global extrema, we use the Closed Interval Method on [-4,4] because the function $f(x)$ is continuous on this interval. The extrema will be two of the numbers: $f(-2), f(3), f(-4), f(4)$, so we make comparisons:

Note that: $f(4) > f(3) > f(2) = 3$ because the definite integrals on $[2, 3]$ and $[3, 4]$ are positive contributions.

$f(-2) < f(2)$ because despite the fact that $f'(x) > 0$ on $(-2, 2)$, the net areas are accumulated from right to left, so they are negative contributions.

The contribution from $x = -2$ to $x = -4$ is positive because the integrand is negative while moving in the negative direction, therefore $f(-4) > f(-2)$. This positive contribution is not large enough to offset and exceed the negative contribution from $x = 2$ to $x = 2$, so the global maximum could not occur at $x = -4$.

Therefore, the global maximum is $f(4)$ and the global minimum is $f(-2)$.

Problem 5

PART A

$$A(R) = \int\limits_1^2 f(x)dx = \int\limits_1^2 \left(\sqrt[3]{x-1}\right) dx = \tfrac{3}{4}(x-1)^{4/3}|_1^2 = \tfrac{3}{4}$$

PART B

$$V_{DISK} = \pi \int\limits_1^2 \left(\sqrt[3]{x-1}\right)^2 dx$$

PART C

$$P = (2-1) + f(2) + L = 2 + \int\limits_1^2 \sqrt{1 + (f'(x))^2}dx$$

PART D

$$\int\limits_2^\infty g(x)dx = \lim_{t\to\infty} \left(\int\limits_2^t g(x)dx \right)$$

$$\int\limits_2^t g(x)dx = \int\limits_2^t e^{-2x+4}dx = \tfrac{-e^{-2x+4}}{2}|_{x=2}^{x=t} = \tfrac{1-e^{-2t+4}}{2}$$

$$\int\limits_2^\infty g(x)dx = \lim_{t\to\infty} \tfrac{1-e^{-2t+4}}{2} = \tfrac{1-0}{2} = \tfrac{1}{2}$$

Problem 6

PART A

When $x = 3$, the geometric series has a ratio of $r = \frac{-3}{5}$ and a first term of $a = 2$. Since the geometric series converges to $\frac{a}{1-r}$, if follows that $g(3) = \frac{2}{1-(-3/5)} = \frac{5}{4}$.

PART B

$f(x) = g'(x) = \frac{-2}{5} + \frac{4}{25}x - \frac{6}{125}x^2 + \frac{8}{625}x^3 + ... + \frac{(2n)(-1)^n}{5^n}x^{n-1} + ...$

PART C

$T_2(x) = \frac{-2}{5} + \frac{4}{25}x - \frac{6}{125}x^2$.

The series is alternating, so the largest error is the absolute value of the next term:

$|\frac{8}{625}(5/2)^3| = \frac{125}{625} = \frac{1}{5}$

Therefore, the largest error in approximation at $x = \frac{5}{2}$ is $\frac{1}{5}$.

PART D

i. $e^x \approxeq 1 + x + \frac{x^2}{2!}$.

ii. $h(x) = 25g(x) - 2e^x = 25\left(2 - \frac{2x}{5} + \frac{2x^2}{25} + ...\right) - 2\left(1 + x + \frac{x^2}{2!} + ...\right) \approxeq 48 - 12x + x^2$