Problem 1

PART A.

$M'(7.5) \approxeq \frac{M(10)-M(5)}{10-5} = \frac{16-7}{5} = \frac{9}{5}$ birds per day per day.

PART B.

i.

$M_3 = 10 * M(5) + 10 * M(15) + 10 * M(25) = 10(7+6+2) = 150$

ii.

Approximately 150 male birds arrived at the nesting area over the 30-day interval. (Alternatively, the net change in the number of male birds present at the nesting area is an increase of 150 birds over the 30-day period.)

PART C.

$$\int_{15}^{45} F(t)dt = \int_{15}^{45} \left(18 + 16\sin(\tfrac{\pi}{20}(t+15))\right) dt = 540 + \tfrac{320}{\pi} \approxeq 641.859$$

Approximately 642 female birds arrive at the nesting area from $t = 15$ to $t = 45$.

PART D.

The functions $F(t)$ and $M(t)$ are differentiable and therefore continuous on the interval $[15, 20]$. The difference function $D(t)$ will therefore be continuous on the interval $[15, 20]$.

We apply the Intermediate Value Theorem:

$D(15) = M(15) - F(15) = 2 > 0$

$D(20) = M(20) - F(20) = 8\sqrt{2} - 13 \approxeq -1.686 < 0$

By the Intermediate Value Theorem, yes, $D(t) = 0$ at least once in interval $15 < t < 20$ since $D(t)$ switches sign in a continuous fashion during this interval.

Problem 2

PART A.

$A(R) = \int\limits_{x=0}^{x=1} g(x)dx \approxeq 1.513$

PART B.

The area of each cross-section is: $A(x) = \text{base} * \text{height} = g(x) * \frac{g(x)}{3} = \frac{(g(x))^2}{3}$

The volume is given by: $\int\limits_{x=0}^{x=1} \left(\frac{(g(x))^2}{3} \right) dx$

PART C.

The point of intersection, using desmos, is $(a, b) = (3.25582, 3.77439)$.

The area of the shaded region is given by:

$A = \int_0^a |f(x) - g(x)|\, dx \approxeq 0.632$

Note that we could have also broken the region into two parts:

$A = \int\limits_0^1 (f(x) - g(x))\, dx + \int\limits_1^a (g(x) - f(x))\, dx \approxeq 0.632$

PART D.

The region T borders the rotational axis (y), so we use the disk method and graphing calculator:

$V_{DISK} = \pi \int\limits_1^{3.5} (h(y))^2\, dy$

Problem 3

PART A.

The given slope field could not describe the differential equation

$$\frac{dH}{dt} = -\frac{1}{15}(H - 20)$$

because for $H > 20$ all the slopes should have a negative value and according to the graph, the line segments are positively-sloped.

PART B.

When $t = 0$, we know that the y-value is $H = 75$. The slope at this point can be measured by the differential equation:

$\frac{dH}{dt}|_{t=0,H=75} = \frac{-1}{15}(75 - 20) = \frac{-55}{15} = \frac{-11}{3}$

The tangent line has equation: $H - 75 = \frac{-11}{3}(t - 0) \rightarrow H(t) = 75 - \frac{11t}{3}$.

PART C.

The second derivative at $t = 0$ has a value of:

$\frac{d^2H}{dt^2}|_{t=0,H=75} = \frac{1}{225}(H - 20)|_{H=75} = \frac{55}{225} > 0$

This implies that the solution curve is concave up at the spot ($t = 0$) where the tangent line is drawn, therefore the approximation at the nearby point ($t = 5$) will be an underestimate.

PART D.

$\frac{1}{H-20}dH = \frac{-1}{15}dt \rightarrow \int \frac{1}{H-20}dH = \int \frac{-1}{15}dt$

$\ln|H - 20| = \frac{-1}{15}t + C$

Plug in $t = 0, H = 75$:

$\ln(55) = C$

$|H - 20| = e^{-t/15} * e^C = 55e^{-t/15} \rightarrow H - 20 = \pm 55e^{-t/15}$

Below we choose the positive option for the right side since $H > 20$.

$H(t) = 20 + 55e^{-t/15}$

Problem 4

PART A

$g(x) = f(x) - \ln x \rightarrow g'(x) = f'(x) - \frac{1}{x}$

$g'(2) = f'(2) - \frac{1}{2} = 1.5 - 0.5 = 1$

PART B

The graph of f has a point of inflection when 1) $f''(x)$, the derivative (slope function) of $f'(x)$ changes sign, 2) $f'(x)$ does not change sign, and 3) $f(x)$ is continuous.

All three conditions are met at $x = 1$.

PART C

f is increasing and concave down whenever $f'(x) > 0$ and $f''(x) < 0$. These conditions hold true on the interval $(1, 3)$

PART D

By FTC: $f(x) - f(2) = \int_2^x f'(t)dt \rightarrow f(x) = f(2) + \int_2^x f'(t)dt = 3 + \int_2^x f'(t)dt$

There are two critical numbers: $f'(x) = 0 \rightarrow x = -2$ or $x = 3$.

To find the global extrema, we use the Closed Interval Method on [-4,4] because the function $f(x)$ is continuous on this interval. The extrema will be two of the numbers: $f(-2), f(3), f(-4), f(4)$, so we make comparisons:

Note that: $f(4) > f(3) > f(2) = 3$ because the definite integrals on $[2, 3]$ and $[3, 4]$ are positive contributions.

$f(-2) < f(2)$ because despite the fact that $f'(x) > 0$ on $(-2, 2)$, the net areas are accumulated from right to left, so they are negative contributions.

The contribution from $x = -2$ to $x = -4$ is positive because the integrand is negative while moving in the negative direction, therefore $f(-4) > f(-2)$. This positive contribution is not large enough to offset and exceed the negative contribution from $x = 2$ to $x = 2$, so the global maximum could not occur at $x = -4$.

Therefore, the global maximum is $f(4)$ and the global minimum is $f(-2)$.

Problem 5

PART A

On the interval that includes $t = 1$, the acceleration is given by:

$a(t) = v'(t) = 4t^3 - 24t^2 + 32t \to a(1) = v'(1) = 4 - 24 + 32 = 12$ feet per second per second.

PART B

$v(1) = 1^4 - 8 * 1^3 + 16 * 1^2 = 1 - 8 + 16 = 9 > 0$ and $a(1) = 12 > 0$.

Since velocity and acceleration have the same sign (both positive), the car is speeding up at $t = 1$.

PART C

On the interval $0 \leq t \leq 4$, velocity is non-negative, so the distance traveled is equal to the displacement:

$\int\limits_0^4 (v(t))\,dt = \int\limits_0^4 (t^4 - 8t^3 + 16t^2)\,dt = \left(\frac{t^5}{5} - 2t^4 + \frac{16t^3}{3}\right)|_{t=0}^{t=4} = 4^3\left(\frac{16}{5} - 8 + \frac{16}{3}\right) = \frac{512}{15}$ feet.

PART D

The average (value of) velocity is given by:

$\frac{1}{12-6}\int\limits_6^{12} v(t)dt = \frac{1}{6}\int\limits_6^{12} (10\cos(\pi t/3) - 10)\,dt = \frac{5}{3}\left(\frac{3}{\pi}\sin(\pi t/3) - t\right)|_{t=6}^{t=12} = \frac{5}{3}(-12+6) = -10$ feet per second.

Problem 6

PART A

Since f is differentiable at $x = 2$, it must also be continuous therefore the limit of $f(x)$ as x approaches 2 must exist and is equal to $f(2)$.

$\lim\limits_{x\to 2} \frac{f(x)}{x} = \frac{\lim\limits_{x\to 2} f(x)}{\lim\limits_{x\to 2} x} = \frac{f(2)}{2} = \frac{3}{2}$

PART B

We apply the Chain Rule: $g(x) = f(f(x)) \rightarrow g'(x) = f'(f(x)) * f'(x)$

$g'(2) = f'(f(2)) * f'(2) = f'(3) * f'(2) = 9 * 4 = 36$

PART C

By FTC:

$h(2) - h(0) = \int\limits_0^2 h'(x)dx = \int\limits_0^2 f'(3x)dx = \frac{1}{3}f(3x)|_{x=0}^{x=2} = \frac{1}{3}(f(6) - f(0)) = \frac{5-(-1)}{3} = 2$

Therefore, $h(2) = h(0) + 2 = 10 + 2 = 12$

PART D

$k(x) = \int\limits_0^x t^2 f(t)dt$

i. Use FTC Part 1:

$k'(x) = x^2 f(x)$

ii.

$k''(x) = x^2 f'(x) + 2xf(x) \rightarrow k''(3) = 9f'(3) + 6f(3) = 9 * 9 + 6 * 8 = 81 + 48 = 129$