Problem 1

a)

$\frac{1}{4-0}\int_0^4 C(t)dt \approx 2.778$ is the average number of acres affected by the invasive species from $t = 0$ to $t = 4$ weeks.

b)

The average rate is $\frac{C(4)-C(0)}{4-0} \approx 1.28201$. Setting this number equal to $C'(t)$, and solving with the graphing calculator, we get $t \approx 2.514$.

c)

$\lim_{t\to\infty} C'(t) = \lim_{t\to\infty} \frac{38}{25+t^2} = \frac{0}{\infty} = 0.$

The rate of change of the number of species approaches zero in the long run.

d)

The domain is $4 \le t \le 36$. The function we are maximizing is:

$$A(t) = C(t) - \int\limits_4^t (0.1\ln(x))\,dx > 0 \rightarrow A'(t) = C'(t) - 0.1\ln(t) \text{ by FTC.}$$

Setting the derivative $A'(t)$ equal to 0 or undefined, we find one critical number in the given domain: $t = 11.442$.

We use the Closed Interval Method with the graphing calculator to evaluate:

$A(4) = 5.128$

$A(36) = 1.743$

$A(11.442) = 7.317$

Therefore, the absolute maximum occurs at $t = 11.442$ weeks.

Problem 2

$r(\theta) = 2(\sin\theta)^2; 0 \le \theta \le \pi.$

a)

$\frac{dr}{d\theta} = 4\sin\theta\cos\theta$

$\left.\frac{dr}{d\theta}\right|_{\theta=1.3} = 4\sin(1.3)\cos(1.3) \approx 1.031$ (Insert graphing calculator model here.)

b)

To find the corresponding angle(s) where the polar curves intersect, set the expressions equal to each other:

$2(\sin\theta)^2 = \frac{1}{2} \to \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

We can use symmetry to double the area in the first quadrant:

$A = 2 \times \left[\frac{1}{2}\int_{\pi/6}^{\pi/2} \left((r(\theta))^2 - \left(\frac{1}{2}\right)^2\right) d\theta\right] \approx 2.067$ (Insert graphing calculator model here.)

Note that the precise value of the area is $\frac{7\sqrt{3}}{16} + \frac{5\pi}{12}$.

c)

Given: $\frac{dx}{d\theta} = 4\sin\theta\cos^2\theta - 2\sin^3\theta$.

To maximize the distance to the y-axis in the first quadrant, it means the same as maximizing $x(\theta)$ for $0 \le \theta \le \pi/2$.

To find critical numbers, we can use technology or see where the algebra takes us:

$\frac{dx}{d\theta} = 2\sin\theta\left[2\cos^2\theta - \sin^2\theta\right] \to \tan^2\theta = 2$ or $\sin\theta = 0$

In the given domain, the only interior candidate is $\theta = \arctan(\sqrt{2}) \approx 0.955317$.

Since $x(0) = x(\pi/2) = 0$, we use the Closed Interval Method to justify that $x(0.955317) \approx 0.770$ is the absolute maximum, so the particle will be farthest from the y-axis when $\theta = \arctan(\sqrt{2}) \approx 0.955$ radian. Alternatively, we could have used the First Derivative Test for Global Extrema to justify.

d)

By the Chain Rule, $\left.\frac{dr}{dt}\right|_{\theta=1.3} = \left.\frac{dr}{d\theta}\right|_{\theta=1.3} \times \left.\frac{d\theta}{dt}\right|_{\theta=1.3} = 1.031 \times 15 = 15.465$ units of distance per unit of time.

Problem 3

a)

$R'(1) \approx \frac{R(2)-R(0)}{2-0} = \frac{100-90}{2-0} = 5$ words per minute per minute.

b)

Yes, by the Intermediate Value Theorem, there must be a value c between 8 and 10 such that $R(c) = 155$ because 155 falls between $R(8) = 150$ and $R(10) = 162$. We can apply IVT here because the function $R(t)$ is differentiable and therefore continuous on the given interval.

c)

$T_3 = \frac{1}{2}[2 \times 190 + 3 \times 250 + 2 \times 312] = 190 + 750 + 312 = 1252$ words.

d)

$\int\limits_0^{10} W(t)dt = \int\limits_0^{10} [-0.3t^2 + 8t + 100]\, dt = (-0.1t^3 + 4t^2 + 100t)\Big|_{t=0}^{t=10} = -0.1 \times 1000 + 4 \times 100 + 100 \times 10 = 1300$ words.

Problem 4

a)

$g'(x) = f(x)$ by the Fundamental Theorem of Calculus,therefore $g'(8) = f(8) = 1$.

b)

$g''(x) = f'(x)$, which changes sign at $x = -3, 3, 6$. These are the inflection points on the graph of g because the second derivative changes sign, the first derivative ($g'(x) = f(x)$) maintains the same sign, and $g(x)$ is continuous.

c)

$g(12) = \frac{1}{2}(6 \times 3) = 9$ (We use geometry here to find the area of the triangle.)

$g(0) = \frac{-9\pi}{2}$ (We use geometry here as well to first find the area of the semi-circle. The negative sign is required because the integrand is positive but the upper bound is less than the lower bound.)

d)

To find the absolute minimum on $[-6, 12]$ we consider critical numbers and endpoints:

$g(-6) = g(6) = 0$

$g(0) = \frac{-9\pi}{2}$

$g(12) = 9$

Since the function $g(x)$ is continuous on the given interval, we use the Closed Interval Method to find that the global minimum occurs at $x = 0$.

Problem 5

$$\frac{dy}{dx} = (3-x)y^2; f(1) = -1$$

a)

After differentiating below, plug in $x = 1, y = -1, \frac{dy}{dx} = 2$.

$\frac{d^2y}{dx^2} = (3-x) \times 2y\frac{dy}{dx} - y^2 \to \frac{d^2y}{dx^2} = 1 \times 2 \times (-1) \times 2 - 1 = -9$

b)

$T_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 = -1 + 2(x-1) - \frac{9}{2!}(x-1)^2$

c)

LEB $= \left|\frac{f'''(z)(x-1)^3}{3!}\right| = \left|\frac{60*(1.1-1)^3}{3!}\right| = \frac{1}{100}$.

$|T_2(1.1) - f(1.1)| < \frac{1}{100}$ The error cannot exceed the Lagrange Error Bound.

d)

We use Euler's Method with $\Delta x = 0.2$ to approximate $f(1.4)$.

Start with: $x_0 = 1, y_0 = -1, m = \left.\frac{dy}{dx}\right|_{x=1,y=-1} = 2$, so the tangent line equation is: $y = -1+2(x-1)$.

Plugging in $x_1 = 1.2$ we get $y_1 = -0.6$.

When $x_1 = 1.2, y_1 = -0.6, m = \left.\frac{dy}{dx}\right|_{x=1,y=-0.6} = \frac{81}{125}$, so the tangent line equation is: $y = \frac{-3}{5} + \frac{81}{125}(x-1.2)$.

Plugging in $x_2 = 1.4$, we get $f(1.4) \approx y_2 = \frac{-3}{5} + \frac{81}{125} \times 0.2 = \frac{81-375}{625} = \frac{-294}{625}$

Problem 6

$$f(x) = \sum_{n=1}^{\infty} \frac{(x-4)^{n+1}}{(n+1)3^n}$$

a)

We perform the Ratio Test to identify the interval of convergence:

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-4)^{n+2}}{(n+2)3^{n+1}} \times \frac{(n+1)3^n}{(x-4)^{n+1}}\right| = \frac{n+1}{n+2}\frac{3^n}{3^{n+1}}|x-4| \to \frac{|x-4|}{3} < 1$

Solve the inequality to find: $1 < x < 7$.

We need to also test the endpoints:

When $x = 1$, the series becomes $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{(n+1)3^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}3}{(n+1)}$, which converges by the Alternating Series Test. Note that the series is a multiple of the Alternating Harmonic Series.

When $x = 7$, the series becomes $\sum_{n=1}^{\infty} \frac{(3)^{n+1}}{(n+1)3^n} = \sum_{n=1}^{\infty} \frac{3}{(n+1)}$, which diverges, since it is a non-zero multiple of the Harmonic Series.

The interval of convergence is $1 \leq x < 7$.

b)

$f'(x) = \frac{x-4}{3} + \frac{(x-4)^2}{3^2} + \frac{(x-4)^3}{3^3} + ... + \frac{(x-4)^n}{3^n} + ...$

c)

The first term of the geometric series is also equal to the ratio: $a = r = \frac{x-4}{3}$. The geometric series converges to:

$\frac{a}{1-r} = \frac{(x-4)/3}{1-(x-4)/3} = \frac{x-4}{3-(x-4)} = \frac{x-4}{7-x}$, as wanted.

d)

The Taylor series for $f'(x)$ does not converge at $x = 8$ because this falls outside of the interval of convergence of the series of the original function $f(x)$.