Problem 1

a)

$\frac{1}{4-0}\int_0^4 C(t)dt \approx 2.778$ is the average number of acres affected by the invasive species from $t = 0$ to $t = 4$ weeks.

b)

The average rate is $\frac{C(4)-C(0)}{4-0} \approx 1.28201$. Setting this number equal to $C'(t)$, and solving with the graphing calculator, we get $t \approx 2.514$ weeks.

c)

$\lim_{t\to\infty} C'(t) = \lim_{t\to\infty} \frac{38}{25+t^2} = \frac{0}{\infty} = 0.$

The rate of change of the number of species approaches zero in the long run.

d)

The domain is $4 \le t \le 36$. The function we are maximizing is:

$$A(t) = C(t) - \int_4^t (0.1\ln(x))\,dx > 0 \to A'(t) = C'(t) - 0.1\ln(t) \text{ by FTC.}$$

Setting the derivative $A'(t)$ equal to 0 or undefined, we find one critical number in the given domain: $t = 11.442$.

We use the Closed Interval Method with the graphing calculator to evaluate:

$A(4) = 5.128$

$A(36) = 1.743$

$A(11.442) = 7.317$

Therefore, the absolute maximum occurs at $t = 11.442$ weeks.

Problem 2

$f(x) = x^2 - 2x; g(x) = x + \sin(\pi x)$

a)

$A = \int\limits_0^3 (g(x) - f(x))\, dx = \frac{9}{2} + \frac{2}{\pi} \approx 5.137.$

b)

The area of a typical cross-section is $B(x) = x(g(x) - f(x))$. The volume is therefore given by:

$V = \int\limits_0^3 B(x)dx = \int\limits_0^3 (x[g(x) - f(x)])\, dx = \frac{27}{4} + \frac{3}{\pi} \approx 7.705.$

c)

$V_{washer} = \pi \int\limits_0^3 ([g(x) + 2]^2 - [f(x) + 2]^2)\, dx$

d)

For the tangent lines to be parallel, we must have $g'(x) = f'(x)$, which is equivalent to $1+\pi\cos(\pi x) = 2x - 2$. Use the graphing calculator to confirm that the solution in the interval $(0, 1)$ is $x = 0.676$.

Problem 3

a)

$R'(1) \approx \frac{R(2)-R(0)}{2-0} = \frac{100-90}{2-0} = 5$ words per minute per minute.

b)

Yes, by the Intermediate Value Theorem, there must be a value c between 8 and 10 such that $R(c) = 155$ because 155 falls between $R(8) = 150$ and $R(10) = 162$. We can apply IVT here because the function $R(t)$ is differentiable and therefore continuous on the given interval.

c)

$T_3 = \frac{1}{2}[2 \times 190 + 3 \times 250 + 2 \times 312] = 190 + 750 + 312 = 1252$ words.

d)

$\int\limits_0^{10} W(t)dt = \int\limits_0^{10} [-0.3t^2 + 8t + 100]\, dt = (-0.1t^3 + 4t^2 + 100t) \Big|_{t=0}^{t=10} = -0.1 \times 1000 + 4 \times 100 + 100 \times 10 = 1300$ words.

Problem 4

a)

$g'(x) = f(x)$ by the Fundamental Theorem of Calculus,therefore $g'(8) = f(8) = 1$.

b)

$g''(x) = f'(x)$, which changes sign at $x = -3, 3, 6$. These are the inflection points on the graph of g because the second derivative changes sign, the first derivative ($g'(x) = f(x)$) maintains the same sign, and $g(x)$ is continuous.

c)

$g(12) = \frac{1}{2}(6 \times 3) = 9$ (We use geometry here to find the area of the triangle.)

$g(0) = \frac{-9\pi}{2}$ (We use geometry here as well to first find the area of the semi-circle. The negative sign is required because the integrand is positive but the upper bound is less than the lower bound.)

d)

To find the absolute minimum on $[-6, 12]$ we consider critical numbers and endpoints:

$g(-6) = g(6) = 0$

$g(0) = \frac{-9\pi}{2}$

$g(12) = 9$

Since the function $g(x)$ is continuous on the given interval, we use the Closed Interval Method to find that the global minimum occurs at $x = 0$.

Problem 5

$x_H(t) = e^{t^2-4t}$

$v_J(t) = 2t(t^2-1)^3$

a)

$x_H(t) = e^{t^2-4t} \rightarrow v_H(t) = (2t-4)e^{t^2-4t}$

Therefore, $v_H(1) = \frac{-2}{e^3}$.

b)

Study the sign of the velocity function for each particle:

Particle H has positive velocity on $(2, 5)$ and negative on $(0, 2)$.

Particle J has positive velocity on $(1, 5)$ and negative on $(0, 1)$.

Therefore, the particles are moving in opposite direction during the interval $1 < t < 2$ since their velocities have opposite signs then.

c)

The velocity of particle J at $t = 2$ is $v_J(2) = 4 > 0$. Its acceleration is also positive, therefore the speed is increasing at $t = 2$. (i.e. the particle is speeding up because velocity and acceleration have the same sign.).

d)

We use FTC: $x_H(2) - x_H(0) = \int_0^2 (2t(t^2-1)^3)\, dt = \frac{(t^2-1)^4}{4}\Big|_{t=0}^{t=2} = \frac{81-1}{4} = 20.$

$x_H(2) = x_H(0) + 20 = 7 + 20 = 27$

Problem 6

$$y^3 - y^2 - y + \frac{1}{4}x^2 = 0$$

a)

$\frac{d}{dx}\left(y^3 - y^2 - y + \frac{1}{4}x^2\right) = \frac{d}{dx}(0)$

$3y^2\frac{dy}{dx} - 2y\frac{dy}{dx} - \frac{dy}{dx} + \frac{x}{2} = 0$

$\frac{dy}{dx}\left(3y^2 - 2y - 1\right) = \frac{-x}{2}$

$\frac{dy}{dx} = \frac{-x}{2(3y^2-2y-1)}$, as wanted.

b)

The slope at the point $(2, -1)$ is $\left.\frac{dy}{dx}\right|_{x=2,y=-1} = \frac{-1}{4}$

The tangent line equation is $y + 1 = \frac{-1}{4}(x - 2)$.

Plug in $x = 1.6$ to get $y = -1 + 0.1 = -0.900 = \frac{-9}{10} \approx y(1.6)$.

c)

The tangent line is vertical when slope is undefined, so we solve for $3y^2-2y-1 = (3y+1)(y-1) = 0$.

The solution in the given domain is $y = 1$.

d)

$2xy + \ln y = 8$.

Implicitly differentiate with respect to time:

$\frac{d}{dt}\left(2xy + \ln y\right) = \frac{d}{dt}(8)$

$2\frac{dx}{dt}y + 2x\frac{dy}{dt} + \frac{1}{y}\frac{dy}{dt} = 0$

Plug in: $x = 4, y = 1, \frac{dx}{dt} = 3$:

$2 \times 3 \times 1 + 2 \times 4 \times \frac{dy}{dt} + 1\frac{dy}{dt} = 0$

$9\frac{dy}{dt} = -6 \to \frac{dy}{dt} = \frac{-2}{3}$ units of distance per unit of time.