Problem 1

a) $C'(5) \approxeq \frac{C(7)-C(3)}{7-3} = \frac{69-85}{4} = -4$ degrees Celcius per minute.

b) $\int_0^{12} C(t)\ dt \approxeq L_3 = f(0)(3-0) + f(3)(7-3) + f(7)(12-7) = 100 * 3 + 85 * 4 + 69 * 5 = 985.$

The expression $\frac{1}{12}\int_0^{12} C(t)\ dt$ is the average temperature of coffee during the 12 minutes.

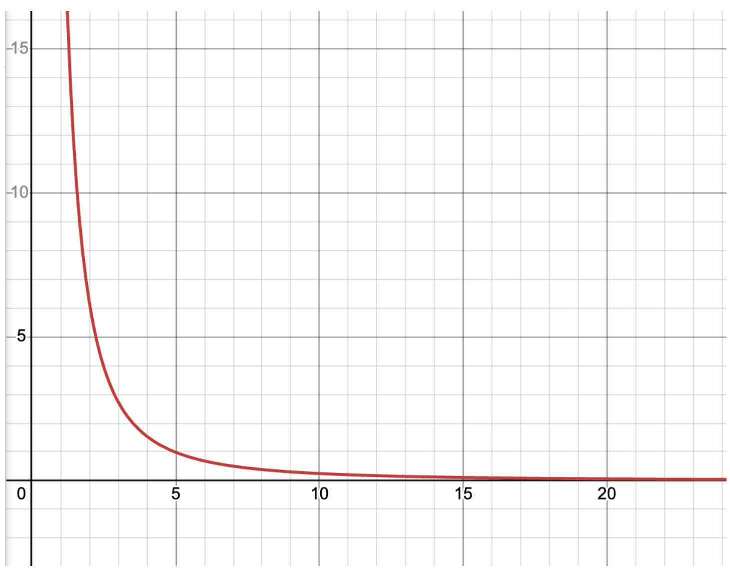
c) By the Fundamental Theorem of Calculus, $\int_{12}^{20} C'(t)\ dt = C(20) - C(12)$.

Therefore, $C(20) = C(12) + \int_{12}^{20} C'(t)\ dt \approxeq= 55 + (-14.6708) \approxeq 40.329$ degrees Celsius.

(Use graphing calculator to approximate the definite integral.)

d) The graph of $C''(t)$ is positive on the interval $12 < t < 20$, so the temperature of coffee is changing at an increasing rate.

$$C''(t) > 0 \rightarrow C'(t) \text{ is increasing.}$$



Problem 2

a)

The speed is given by: $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

Plug in $t = 2$ to get: speed $\approxeq 12.305$. (use graphing calculator)

b)

Total distance is given by:

$\int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \approxeq 15.902$

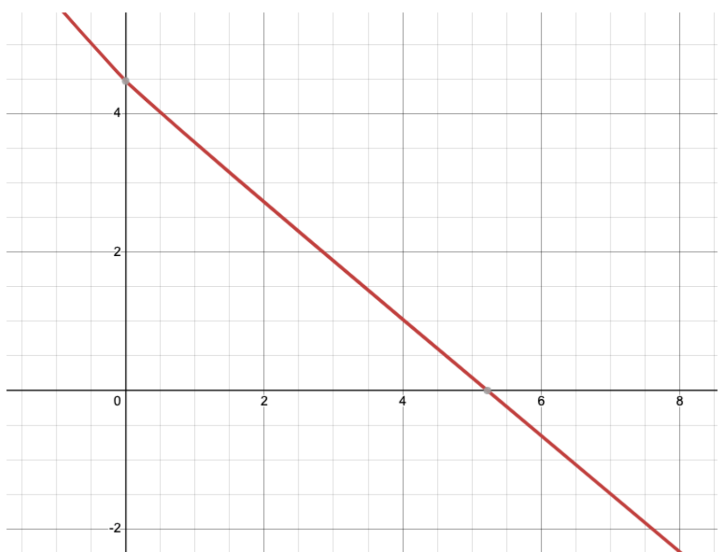
c)

By the Fundamental Theorem of Calculus:

$y(2) - y(0) = \int_0^2 y'(t)$ dt, so it follows that $y(0) = y(2) - \int_0^2 y'(t)$ dt $\approxeq 6 - 7.1736 \approxeq -1.174$

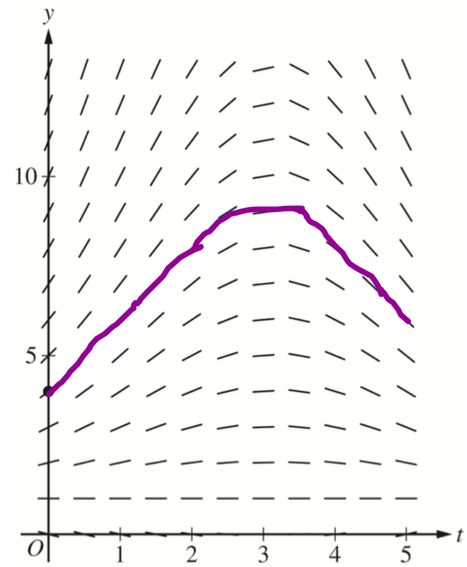
d)

If located in the first quadrant, the particle is moving towards the $x-$axis whenever $y'(t) < 0$. We use the graphing calculator with a window of $2 \leq t \leq 8$ to determine that the $y-$coordinate is decreasing for $5.222 < t \leq 8$, which means the particle is moving towards the $x-$axis during this time interval. A graph of $y'(t)$ is shown below.



Problem 3

a) Below is a solution curve that passes through the point (0,4).



b) A critical number occurs when $\frac{dH}{dt} = 0$ or is undefined. Because $H(t) > 1$, it follows that $\cos\left(\frac{t}{2}\right) = 0$. Therefore, $\frac{t}{2} = \frac{\pi}{2} \rightarrow t = \pi$ is the solution we are looking for in the interval $0 < t < 5$. We need to study the sign change:

For $t < \pi \rightarrow \cos\left(\frac{t}{2}\right) > 0 \rightarrow \frac{dH}{dt} > 0$.

For $\pi < t \rightarrow \cos\left(\frac{t}{2}\right) < 0 \rightarrow \frac{dH}{dt} < 0$.

By the first derivative test for local extrema, the depth water has reached a local maximum at the given critical number. Note that we ignored the factor $(H - 1)$ when studying the sign since we know that $H(t) > 1$.

c) We first separate the variables before integrating:

$\frac{1}{H-1} dH = \frac{1}{2} \cos\left(\frac{t}{2}\right) dt$

$\int \frac{1}{H-1} dH = \int \frac{1}{2} \cos\left(\frac{t}{2}\right) dt$

$\ln |H - 1| = \sin\left(\frac{t}{2}\right) + C \leftarrow H(0) = 4$

$C = \ln 3$ so we have: $|H(t) - 1| = e^{\sin(t/2)} e^{\ln 3} = 3e^{\sin(t/2)}$

$H(t) = 1 \pm 3e^{\sin(t/2)}$. The option that satisfies our initial condition is: $H(t) = 1 + 3e^{\sin(t/2)}$.

Problem 4

a) $g(x) = \int_0^x f(t)\ dt$.

$g(-6) = \int_0^{-6} f(t)\ dt = -\int_{-6}^{0} f(t)\ dt = -12$.

$g(4) = \int_0^4 f(t)\ dt = \frac{base*height}{2} = \frac{4*2}{2} = 4$ (the area of the right triangle in the first quadrant).

$g(6) = \int_0^6 f(t)\ dt = \int_0^4 f(t)\ dt + \int_4^6 f(t)\ dt = \frac{4*2}{2} - \frac{2*1}{2} = 3$.

b) The graph of $g(x)$ has a critical point whenever its derivative, $g'(x) = f(x)$ (by FTC) is zero or undefined. On the given interval, the only such number is $x = 4$, since $f(4) = 0$.

c) $h(x) = \int_{-6}^{x} f'(t)\ dt$. We use the Fundamental Theorem of Calculus in the computations below.

$h(6) = \int_{-6}^{6} f'(t)dt = f(6) - f(-6) = (-1) - (0.5) = -\frac{3}{2}$.

$h'(x) = f'(x)$, therefore $h'(6) = f'(6) = \frac{rise}{run} = -\frac{1}{2}$.

$h''(x) = f''(x)$. Since the graph of f is a linear function, it follows that its second derivative is zero everywhere. Therefore, $h''(6) = 0$.

Problem 5

a) By the Fundamental Theorem of Calculus:

$h(x) = \int_0^x \sqrt{1 + (f'(t))^2}\ dt \rightarrow h'(x) = \sqrt{1 + (f'(x))^2}$

Therefore, $h'(\pi) = \sqrt{1 + 6^2} = \sqrt{37}$.

b) The integral $\int_0^\pi \sqrt{1 + (f'(x))^2}\ dx$ measures the arclength of the graph of $y = f(x)$ on the interval $[0, \pi]$.

c)

At $x_0 = 0, y_0 = 0, \text{slope} = 5$

The tangent line at $(0, 0)$ has slope 5, so its equation is $y = 5x$. Plugging in $x_1 = \pi$, we get $y_1 = 5\pi$.

At $x_1 = \pi, y_1 = 5\pi, \text{slope} = 6$

The tangent line at $(\pi, 5\pi)$ has slope 6, so its equation is $y = 5\pi + 6(x - \pi)$. Plugging in $x_2 = 2\pi$, we get $y_2 = 11\pi$.

Using Euler's method, we conclude that $f(2\pi) \approxeq 11\pi$.

d) $\int (t + 5) \cos\left(\frac{t}{4}\right)\ dt$

Use integration by parts:

$u = t + 5 \rightarrow u' = 1$

$v' = \cos\left(\frac{t}{4}\right) \rightarrow v = 4\sin\left(\frac{t}{4}\right)$

$\int (t + 5) \cos\left(\frac{t}{4}\right)\ dt = 4(t + 5)\sin\left(\frac{t}{4}\right) + 16 \int \frac{-1}{4} \sin\left(\frac{t}{4}\right)\ \text{dt}\ = 4(t + 5)\sin\left(\frac{t}{4}\right) + 16\cos\left(\frac{t}{4}\right) + C$

Problem 6

a)

When $x = 6$, the Maclaurin series for f becomes

$$\sum_{n=1}^{\infty} \frac{(n+1)6^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{(n+1)}{n^2}$$

We can use the Limit Comparison Test with the harmonic series $\sum \frac{1}{n}$, so that $a_n = \frac{n+1}{n^2}$ and $b_n = \frac{1}{n}$.

$$\frac{a_n}{b_n} = \frac{\frac{n+1}{n^2}}{\frac{1}{n}} = \frac{n+1}{n} \to 1 > 0$$

Because the limit of the ratio is a finite positive number, the given series behaves the same as the harmonic series, which means they both diverge.

b)

The maximum error is the absolute value of the next term, so we plug in $n = 4$:

$$\text{error bound} = \left| \frac{5}{16} \frac{(-1)^4}{2^4} \right| = \frac{5}{256} < \frac{5}{250} = \frac{1}{50}, \text{ as wanted.}$$

The error $|f(-3) - S_3|$ is therefore less than $\frac{1}{50}$.

c)

$$f(x) = \sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n} \to f'(x) = \sum_{n=1}^{\infty} \frac{n(n+1)x^{n-1}}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{(n+1)x^{n-1}}{n 6^n}$$

The radius of convergence of the Maclaurin series for f' must be 6, the same as that of the original function. Although not necessary, we can confirm by using the Ratio Test:

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+2)x^n}{(n+1)6^{n+1}} \frac{n6^n}{(n+1)x^{n-1}} \right| \to \left| \frac{x}{6} \right| < 1$$

This confirms that the radius is 6 ($|x| < 6$).

d) Use the Ratio Test:

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+2)x^{2(n+1)}}{(n+1)^2 3^{n+1}} \frac{n^2 3^n}{(n+1)x^{2n}} \right| \to \left| \frac{x^2}{3} \right| < 1$$

This is equivalent to $x^2 < 3$, so the radius of convergence is $\sqrt{3}$.