ap-calc.github.io | AP calculus AB | 2023 FR Notes

These notes likely contain typos and mistakes. This document will be updated with corrections. Last updated: May 10, 2023

Problem 1

a)

The integral measures the amount of gasoline (in gallons) pumped into the tank during the time interval [60, 135] (seconds).

$R_3 = (90-60)\cdot f(90) + (120-90)\cdot f(120) + (135-120)\cdot f(135) = 30\cdot 0.15 + 30\cdot 0.1 + 15\cdot 0.05 = 8.250$ gallons

b)

By Rolle's theorem, since *f(t)* is differentiable on (60, 120), *f(t) is* continuous on [60, 120], and *f(60)=0.1=f(120)*, it follows that at least once *f′(c)=0* for some *c* in (60, 120).

c)

$avg\ rate = \dfrac{1}{(150-0)}\displaystyle\int_0^{150} g(t)\,\mathrm{d}t \approx 0.096\ gallons\ per\ second$ (Graphing calculator)

d)

$g'(140) \approx -0.005\ gallons\ per\ second\ per\ second$ (Graphing calculator)

g′(140) is negative. This means that the rate at which the gasoline flows is decreasing ten seconds before the task of filling the tank ends.

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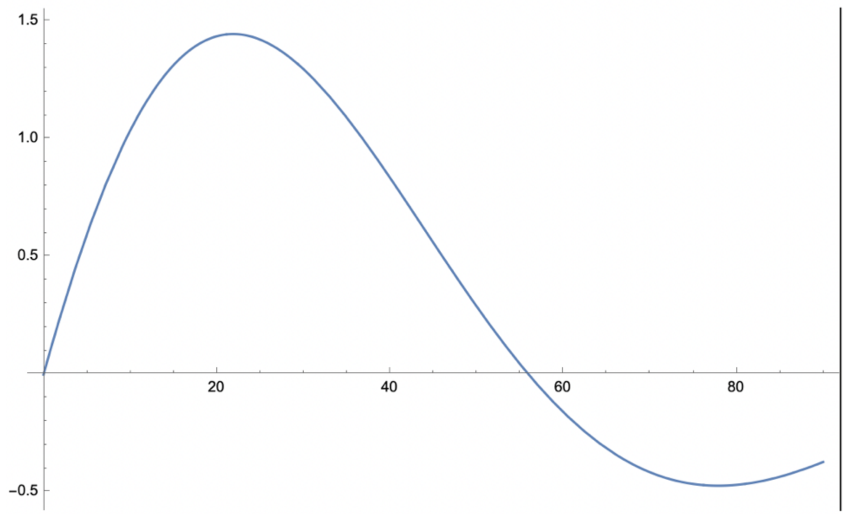
Problem 2

a)

Stephen changes direction at *t=56 seconds* because the velocity changes sign at that time:



b)

$$a(60) = v'(60) \approx -0.036 \textit{ meters per second per second } (TI-84)$$

$$v(60) \approx -0.160 \textit{ meters per second } (TI-84)$$

Since velocity and acceleration are both negative, Stephen is speeding up at *t=60* seconds.

c)

The difference in position is measured by displacement. Using FTC:

$$s(80) - s(20) = \int_{20}^{80} v(t)\,dt \approx 23.384 \textit{ meters } (TI-84)$$

d)

The total distance on the interval *[0, 90]* seconds is given by:

$$\int_{0}^{90} |v(t)|\,dt \approx 62.164 \textit{ meters } (TI-84)$$

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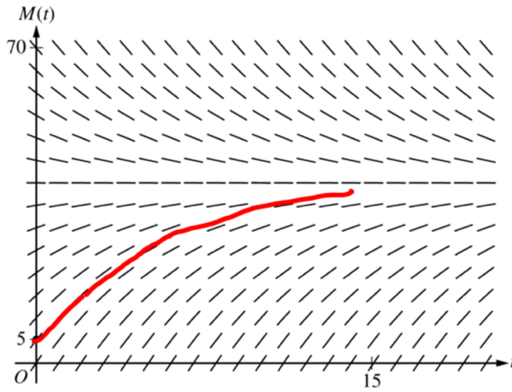
Problem 3

a)

A solution that passes through the point (0, 5) is shown below.



b)

$$L(t) - M(0) = \frac{dM}{dt}_{t=0}(t-0)$$

$$L(t) - 5 = \frac{1}{4}(40-5)\cdot t$$

$$L(t) = 5 + \frac{35}{4}t$$

$$M(2) \approx L(2) = 5 + \frac{70}{4} = 22.500 \textit{ degrees Celcius}$$

c)

$$\frac{d^2M}{dt^2} = \frac{d}{dt}\left(\frac{dM}{dt}\right) = \frac{d}{dt}\left(10 - \frac{M}{4}\right) = \left(\frac{-1}{4}\right)\frac{dM}{dt} = \left(\frac{-1}{4}\right)\left(\frac{1}{4}(40-M)\right) = \frac{M-40}{16}$$

Since *M(t)* is less than 40 for all values of *t,* the second derivative must be negative, which implies that the solution curve *M(t)* will be concave down. The tangent approximation in part b) is therefore an overestimate.

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d)

$$\frac{dM}{dt} = \frac{1}{4}(40 - M)$$

$$\frac{1}{40 - M} dM = \frac{1}{4} dt$$

$$\int \frac{1}{40 - M} dM = \int \frac{1}{4} dt$$

$$-\ln|40 - M| = \frac{t}{4} + C \rightarrow -\ln 35 = C$$

$$\ln|40 - M| = -\frac{t}{4} + \ln(35)$$

$$|40 - M| = 35 \cdot e^{-\frac{t}{4}} \rightarrow 40 - M = \pm 35e^{-\frac{t}{4}} \rightarrow M(t) = 40 - 35e^{-\frac{t}{4}}$$

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Problem 4

a)

The graph of *f′* does not change sign at *x=6*, therefore the graph of *f* does not have a relative extremum there.

b)

The graph of *f* will be concave down whenever its second derivative is negative, or whenever the slope of the graph of *f′* is negative. This holds true on the intervals *(-2, 2)* and *(4, 6)*.

c)

Since *f(x)* is continuous, its graph approaches 1 as *x* approaches 2. Since the numerator and denominator both go to zero, we use L'Hospital's Rule below:

$$\lim_{x \to 2} \frac{6f(x) - 3x}{x^2 - 5x + 6} = \lim_{x \to 2} \frac{6f'(x) - 3}{2x - 5} = \frac{6 \cdot 0 - 3}{4 - 5} = 3$$

d)

We consider the values of *f(x)* at the endpoints as well as at the critical points in the interior. We use geometry (triangles and quarter circles) and FTC to evaluate the net areas:

$$f(2) = 1$$

$$f(8) = f(2) + \frac{2 \cdot 2}{2} + \left(2 \cdot 4 - \frac{\pi 2^2}{2}\right) = 11 - 2\pi$$

$$f(-1) = f(2) + \frac{3 \cdot 2}{2} = 4$$

$$f(-2) = f(-1) - \frac{1 \cdot 2}{2} = 4 - 1 = 3$$

Using the Closed Interval Method, *f(2)=1* is the absolute minimum.

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Problem 5

a)

$h'(x) = f'(g(x))\,g'(x) \rightarrow h'(7) = f'(0) \cdot g'(7) = 12$

b)

$k''(x) = 2f(x)f'(x)\,g(x) + (f(x))^2 g'(x)$

$k''(4) = 2 \cdot f(4)f'(4)\,g(4) + (f(4))^2 g'(4)$

$k''(4) = 2 \cdot 4 \cdot 3 \cdot (-3) + 4^2 \cdot 2 = -72 + 32 = -40 < 0$

Since the second derivative is negative, the graph of *k(x)* is concave down at *x=4.*

c)

We use FTC in the computation that follows:

$m(2) = 5 \cdot 2^3 + f(2) - f(0) = 40 + 7 - 10 = 37$

d)

We use FTC to find the derivative:

$m'(x) = 15x^2 + f'(x)$

$m'(2) = 15 \cdot 4 + f'(2) = 60 - 8 = 52 > 0$

Since the slope is positive, the graph of *m(x)* is increasing at *x=2*.

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Problem 6

a)

We differentiate implicitly:

$$6y + 6x\frac{dy}{dx} = 3y^2\frac{dy}{dx} \to \frac{dy}{dx}\left(6x - 3y^2\right) = -6y$$

$$\frac{dy}{dx} = \frac{-6y}{-3\left(y^2 - 2x\right)} = \frac{2y}{y^2 - 2x}$$

b)

The tangent line will be horizontal whenever the first derivative is zero. We investigate by setting the numerator equal zero. This leads to an impossible condition, so no such point exists.

$$\frac{dy}{dx} = 0 \to 2y = 0 \to y = 0$$

$$6x \cdot 0 = 2 + 0^3$$

$$0 = 2$$

c)

To have a vertical tangent line, the denominator of the first derivative must equal zero (while the numerator is nonzero).

$$y^2 = 2x \Leftrightarrow x = \frac{y^2}{2}$$

$$6\left(\frac{y^2}{2}\right)y = 2 + y^3 \Leftrightarrow 3y^3 = 2 + y^3 \Leftrightarrow y^3 = 1$$

$$y = 1$$

$$x = \frac{1}{2}$$

$$\left(\frac{1}{2}, 1\right)$$

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d)
We differentiate implicitly with respect to time *t*.

$$6\left(x\frac{dy}{dt}+\frac{dx}{dt}y\right)=3y^2\frac{dy}{dt}$$

$$2\left(x\frac{dy}{dt}+\frac{dx}{dt}y\right)=y^2\frac{dy}{dt}$$

$$2\left(\frac{1}{2}\cdot\frac{dy}{dt}+\frac{2}{3}\cdot(-2)\right)=(-2)^2\cdot\frac{dy}{dt}$$

$$-\frac{8}{3}=3\cdot\frac{dy}{dt}\rightarrow\frac{dy}{dt}=-\frac{8}{9}\textit{unit per second}$$