Problem 1

Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).

(a) How many fish enter the lake over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ? Give your answer to the nearest whole number.

(b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ?

(c) At what time t, for $0 \le t \le 8$, is the greatest number of fish in the lake? Justify your answer.

(d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ($t = 5$) ? Explain your reasoning.

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a)

$$\int_0^5 E(t)\,dt = \int_0^5 20 + 15\sin\left(\frac{\pi t}{6}\right)dt \approx 278.192$$

The number of fish that enter the lake during the five-hour interval is approximately 278.

b)

$$\frac{1}{5-0}\int_0^5 L(t)\,dt = \frac{1}{5}\int_0^5 4 + 2^{0.1t^2}\,dt \approx 6.059$$

The average number of fish that leave the lake during the five-hour interval from midnight till 5AM is approximately 6.059 fish/hour.

c)
Using the Closed Interval Method, we first find critical numbers by setting the two rates equal to each other. No information is given about the original number of fish present in the lake, so we will call this amount *Q(0).*

$$Q(x) = Q(0) + \int_0^x E(t) - L(t)\,dt$$

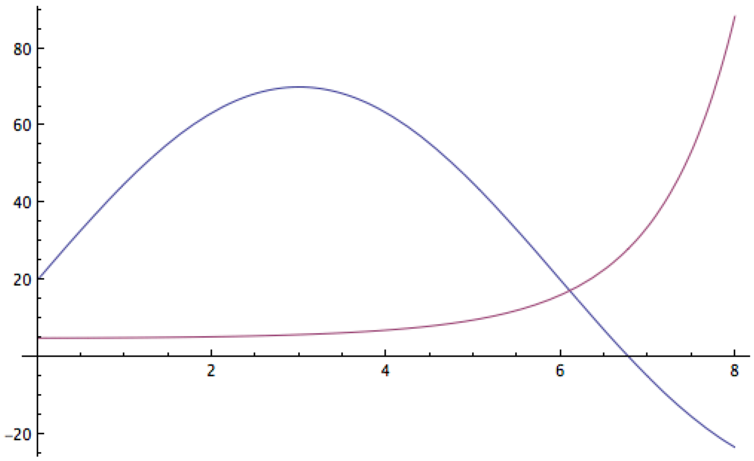
$$Q'(x) = E(x) - L(x) = 0 \ or \text{ undefined}$$

$$L(t) = E(t) \rightarrow t \approx 6.2035644 = m$$

$$Q(m) = Q(0) + \int_0^m E(t) - L(t)\,dt = Q(0) + 268.326$$

$$Q(8) = Q(0) + \int_0^8 E(t) - L(t)\,dt = Q(0) + 181.188$$

$$t < m \rightarrow Q'(x) = E(x) - L(x) > 0$$
$$t > m \rightarrow Q'(x) = E(x) - L(t) < 0$$
$$\therefore Q(m) \rightarrow \text{local (and global) max}$$



Using the Closed Interval Method (or the First Derivative Test for Global Extrema), at approximately 6.204 hours, the number of fish in the lake is the greatest.

d)
We compare E(5) versus L(5).

$$E(5) \approx 45.000 \; fish / hr$$
$$L(5) \approx 9.657 \; fish / hr$$

At 5AM, fish are entering the lake at a higher rate than they are leaving. Therefore, the overall number of fish in the lake is increasing at 5AM.

Problem 2

t (hours)	0	0.3	1.7	2.8	4
$v_P(t)$ (meters per hour)	0	55	–29	55	48

The velocity of a particle, P, moving along the x-axis is given by the differentiable function v_P, where $v_P(t)$ is measured in meters per hour and t is measured in hours. Selected values of $v_P(t)$ are shown in the table above. Particle P is at the origin at time $t = 0$.

(a) Justify why there must be at least one time t, for $0.3 \le t \le 2.8$, at which $v_P'(t)$, the acceleration of particle P, equals 0 meters per hour per hour.

(b) Use a trapezoidal sum with the three subintervals [0, 0.3], [0.3, 1.7], and [1.7, 2.8] to approximate the value of $\int_0^{2.8} v_P(t)\, dt$.

(c) A second particle, Q, also moves along the x-axis so that its velocity for $0 \le t \le 4$ is given by $v_Q(t) = 45\sqrt{t}\cos\left(0.063t^2\right)$ meters per hour. Find the time interval during which the velocity of particle Q is at least 60 meters per hour. Find the distance traveled by particle Q during the interval when the velocity of particle Q is at least 60 meters per hour.

(d) At time $t = 0$, particle Q is at position $x = -90$. Using the result from part (b) and the function v_Q from part (c), approximate the distance between particles P and Q at time $t = 2.8$.

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a)
Since the velocity function is differentiable (and therefore continuous), we can apply Rolle's Theorem on the interval [0.3, 2.8]. At least once,

$$a(c) = v_P'(c) = \frac{v(2.8) - v(0.3)}{2.8 - 0.3} = \frac{55 - 55}{2.5} = 0$$

b)

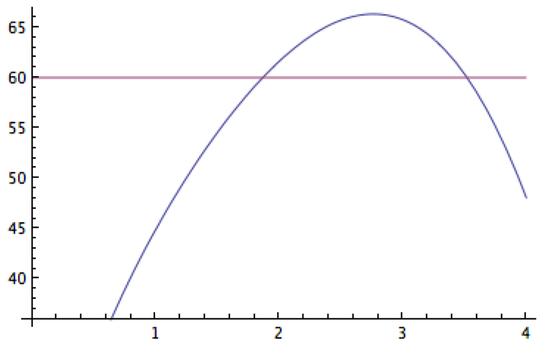
$$\int_0^{2.8} v_p(t)\,dt \approx \frac{(55+0)*0.3}{2} + \frac{(-29+55)*1.4}{2} + \frac{(-29+55)*1.1}{2} = 40.750$$

The displacement of the particle during the interval [0, 2.8] is approximately 40.750 meters.

c)
Using the graphing calculator, we find the velocity to be above 60 meters per hour during the interval [1.8661815, 3.5191744]

The total distance traveled by particle Q during the above interval is given by:

$$\int_{1.8661815}^{3.5191744} \left|v_Q(t)\right| dt \approx 106.1087505 \approx 106.109\, meters$$



d)
We use the Fundamental Theorem of Calculus to determine the position of each particle:

$$x_Q(2.8) = x_Q(0) + \int_0^{2.8} v_Q(t)\,dt$$

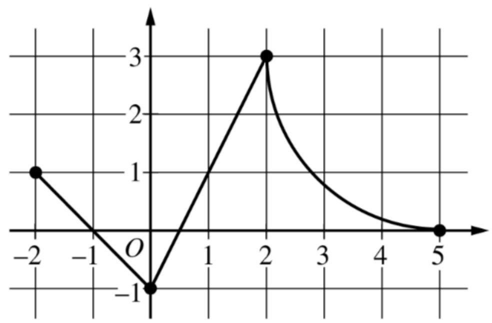
$$x_Q(2.8) = -90 + \int_0^{2.8} 45\sqrt{t}\cos(0.063t^2)\,dt$$

$$x_Q(2.8) \approx 45.9377$$

$$x_P(2.8) \approx x_P(0) + 40.750 = 40.750$$

$$|\,x_Q(2.8) - x_P(2.8)\,| \approx 5.188\, meters$$

Problem 3



Graph of f

The continuous function f is defined on the closed interval $-6 \le x \le 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3 - \sqrt{5})$ is on the graph of f.

(a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

(b) Evaluate $\int_{3}^{5} (2f'(x) + 4)\,dx$.

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

(d) Find $\lim\limits_{x \to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$.

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a)

$$\int_{-6}^{-2} f(x)\,dx = \int_{-6}^{5} f(x)\,dx - \int_{-2}^{5} f(x)\,dx =$$

$$= 7 - \left(\int_{-2}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx \right) =$$

$$= 7 - \left(0 + 0 + \frac{(3+1)*1}{2} + \left(9 - \frac{9\pi}{4} \right) \right) = -4 + \frac{9\pi}{4} = \frac{9\pi - 16}{4}$$

b)

$$\int_{3}^{5} 2f'(x) + 4\,dx = 2\int_{3}^{5} f'(x)\,dx + \int_{3}^{5} 4\,dx = 2\big(f(5) - f(3)\big) + 8 =$$

$$= 2\left(0 - \left(3 - \sqrt{5}\right)\right) + 8 = 2 + 2\sqrt{5}$$

c)

We use the Closed Interval Method:

$$g(x)=\int_{-2}^{x} f(t)\,dt \to g'(x)=f(x)=0$$

$$\to x=-1, x=0.5, x=5$$

$$g(-2)=0$$

$$g(-1)=0.500$$

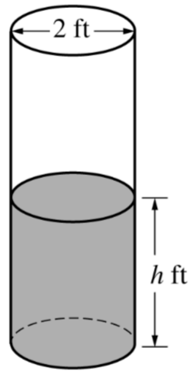
$$g(0.5)=\frac{0.5*(-1)}{2}=-0.250$$

$$g(5)=2+9-\frac{9\pi}{4}=\frac{44-9\pi}{4}\to \text{Absolute Maximum.}$$

d)

$$\lim_{x\to 1}\frac{10^x-3f'(x)}{f(x)-\arctan(x)}=\frac{10-3*2}{1-\frac{\pi}{4}}=\frac{4}{\frac{4-\pi}{4}}=\frac{16}{4-\pi}$$

Problem 4



A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

(b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

(c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.

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a)

$$r = 1$$

$$V = \pi r^2 h = \pi h \rightarrow \frac{dV}{dt} = \pi \frac{dh}{dt}$$

$$\frac{dV}{dt} = \pi\left(\frac{-1}{10}\sqrt{h}\right) = \frac{-\pi\sqrt{h}}{10}$$

$$\left.\frac{dV}{dt}\right|_{h=4} = \frac{-\pi\sqrt{4}}{10} = \frac{-\pi}{5}\, ft^3 / s$$

When the water is 4 feet deep, the volume of water in the barrel is decreasing at a rate of $\pi / 5$ cubic feet per second.

b)

$$\frac{d}{dt}\left(\frac{dh}{dt}\right)=\frac{-1}{10}*\frac{1}{2\sqrt{h}}*\frac{dh}{dt}=\frac{-1}{20\sqrt{h}}\frac{-\sqrt{h}}{10}=\frac{1}{200}>0$$

The rate of change of the height of water with respect to time is increasing.

c)

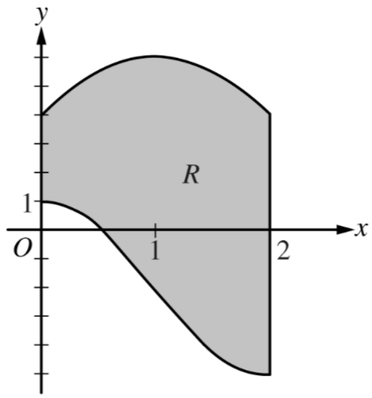
$$\frac{dh}{dt}=\frac{-\sqrt{h}}{10}$$

$$\frac{1}{\sqrt{h}}dh=\frac{-1}{10}dt\rightarrow\int\frac{1}{\sqrt{h}}dh=\int\frac{-1}{10}dt$$

$$2\sqrt{h}=\frac{-t}{10}+C\rightarrow 2\sqrt{5}=C$$

$$\sqrt{h}=\frac{-t}{20}+\sqrt{5}\rightarrow h(t)=\left(\sqrt{5}-\frac{t}{20}\right)^2$$

Problem 5



Let R be the region enclosed by the graphs of $g(x) = -2 + 3\cos\left(\frac{\pi}{2}x\right)$ and $h(x) = 6 - 2(x-1)^2$, the y-axis, and the vertical line $x = 2$, as shown in the figure above.

(a) Find the area of R.

(b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis has area $A(x) = \frac{1}{x+3}$. Find the volume of the solid.

(c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 6$.

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a)

$$A(R)=\int_0^2 h(x)-g(x)dx=\int_0^2\left(6-2(x-1)^2\right)-\left(-2+3\cos\left(\frac{\pi x}{2}\right)\right)dx$$

$$=\left(8x-\frac{2(x-1)^3}{3}-\frac{6}{\pi}\sin\left(\frac{\pi x}{2}\right)\right)\Bigg|_{x=0}^{x=2}=\frac{44}{3}$$

b)

$$V=\int_{x=0}^{x=2} A(x)dx=\int_{x=0}^{x=2}\frac{1}{x+3}dx=\left(\ln|x+3|\right)\Big|_{x=0}^{x=2}=\ln 5-\ln 3=\ln\left(\frac{5}{3}\right)$$

c)

$$V_{WASHER}=\pi\int_{x=0}^{x=2} R^2-r^2\,dx=\int_{x=0}^{x=2}\left(6-g(x)\right)^2-\left(6-h(x)\right)^2dx=$$

$$=\pi\int_{x=0}^{x=2}\left(6-\left(-2+3\cos\left(\frac{\pi}{2}x\right)\right)\right)^2-\left(6-\left(6-2(x-1)^2\right)\right)^2dx$$

Problem 6

Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x - 2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

(a) Find $h'(2)$.

(b) Let a be the function given by $a(x) = 3x^3 h(x)$. Write an expression for $a'(x)$. Find $a'(2)$.

(c) The function h satisfies $h(x) = \dfrac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$. It is known that $\lim_{x\to 2} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x\to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

(d) It is known that $g(x) \le h(x)$ for $1 < x < 3$. Let k be a function satisfying $g(x) \le k(x) \le h(x)$ for $1 < x < 3$. Is k continuous at $x = 2$? Justify your answer.

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a)

$$h'(2) = slope = \frac{2}{3}$$

b)

$$a(x) = 3x^3 h(x) \rightarrow a'(x) = 3x^3 h'(x) + 9x^2 h(x)$$

$$a'(2) = 24h'(2) + 36h(2) = 24 * \frac{2}{3} + 36 * 4 = 16 + 144 = 160$$

c)

In order for L'Hospital's Rule to be applicable, we must have an indeterminate form, so the denominator and numerator both have to equal zero when x = 2. Therefore: $f(2) = 1$.

Since h(x) is continuous, its limit as x approaches 2 must equal the y-value when x=2.

$$h(2) = \lim_{x\to 2} \frac{x^2 - 4}{1 - \left(f(x)\right)^3} = \lim_{x\to 2} \frac{2x}{-3\left(f(x)\right)^2 f'(x)}$$

$$h(2) = \frac{4}{-3\left(f(2)\right)^2 f'(2)} = 4 \rightarrow -3 * 1 * f'(2) = 1$$

$$f'(2) = \frac{-1}{3}$$

d)

Assuming k(x) is defined at x=2, by the Squeeze Theorem (conditions below), k(x) must be continuous at x=2.

$$g(x) \le k(x) \le h(x)$$

$$\lim_{x\to 2} g(x) = 4 = \lim_{x\to 2} h(x)$$