Draft Solutions: AP Calculus BC | Free Response 2017 | Shubleka

Problem 1

a)

$L_3 = (2 - 0)\ 50.3 + (5 - 2)*14.4 + (10 - 5)*6.5 = 176.300$ cubic feet

Using a left hand sum with three subintervals, the volume of the tank is approximately 176.300 cubic feet.

b)

The table suggests that $A(h)$ is a decreasing function, so the left hand sum will be an overestimate of the true volume of the tank.

c)

$$\int_0^{10} f(h)\,dh = \int_0^{10} \frac{50.3}{e^{0.2h} + h}\,dh = 101.325 \text{ cubic feet}$$

d)

$$V(h) = V(0) + \int_0^{h} f(z)\,dz \rightarrow \frac{dV}{dh} = f(h) \quad (by\, FTC)$$

$$\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} = f(h)\frac{dh}{dt} = f(5) * 0.26 = 1.694 \frac{cf}{\min}$$

When the height of the water is 5 feet, the volume of water is increasing at a rate of 1.694 cubic feet per minute.

Problem 2

a)

$$A=\frac{1}{2}\int_0^{\frac{\pi}{2}}\left(f(\theta)\right)^2\,d\theta=\frac{1}{2}\int_0^{\frac{\pi}{2}}\left(1+\sin\theta\cos(2\theta)\right)^2\,d\theta=0.648$$

b)

$$\frac{1}{2}\int_0^{k}\left(g(\theta)\right)^2-\left(f(\theta)\right)^2\,d\theta=\frac{1}{2}\int_k^{\frac{\pi}{2}}\left(g(\theta)\right)^2-\left(f(\theta)\right)^2\,d\theta$$

c)

$$w(\theta)=g(\theta)-f(\theta)$$

$$w_A=\frac{1}{\frac{\pi}{2}-0}\int_0^{\pi/2}g(\theta)-f(\theta)\,d\theta=\frac{2}{\pi}\int_0^{\pi/2}g(\theta)-f(\theta)\,d\theta=0.485$$

d)

$$g(\theta)-f(\theta)=0.485\rightarrow\theta=0.518$$

$$w'(\theta)=g'(\theta)-f'(\theta)$$

$$w'(0.518)=g'(0.518)-f'(0.518)=-0.581<0$$

The function that measures the distance is decreasing at the angle when the distance between the curves is equal to the average distance.

Problem 3

a)

$$f(-2)-f(-6)=\int_{-6}^{-2} f(t)\,dt$$

$$7-f(-6)=\frac{2*4}{2}=4 \rightarrow f(-6)=3$$

$$f(5)-f(-2)=\int_{-2}^{5} f(t)\,dt$$

$$f(5)=3+\left(\frac{3*2}{2}-\frac{\pi 2^2}{2}\right)=6-2\pi$$

b)

f is increasing whenever its derivative is positive. This occurs on the intervals (-6, -2) and (2, 5).

c)

We consider the critical points at x=-2, x=2, as well as the endpoints, which we evaluated in part a).

$$f(-6)=3$$

$$f(-2)=7$$

$$f(2)=7-2\pi$$

$$f(5)=6-2\pi \rightarrow \text{absolute minimum}$$

d)

$$f''(-5)=\frac{-2}{4}=\frac{-1}{2}$$

$$f''(3)=DNE\,(corner)$$

Problem 4

a)

$$H'(0) = \frac{-1}{4}(91-27) = -16$$

$$y - 91 = -16(t-0) \rightarrow y = -16t + 91$$

$$y(3) = -16*3 + 91 = 43$$

Using the tangent line approximation, the temperature at t = 3 is approximately 43 degrees Celcius.

b)

$$H''(t) = \frac{-1}{4}\frac{dH}{dt} = \left(\frac{-1}{4}\right)\left(\frac{-1}{4}\right)(H-27) = \frac{1}{16}(H-27)$$

$H'' > 0$ (since H is always greater than 27)

$\rightarrow$ The graph is concave up, therefore the linear approximation is an underestimate.

c)

$$\frac{dG}{dt} = -(G-27)^{2/3} \quad G(0) = 91$$

$$(G-27)^{-2/3}\,dG = -1\,dt$$

$$\int (G-27)^{-2/3}\,dG = \int -1\,dt$$

$$3*(G-27)^{1/3} = -t + C$$

$$3(91-27)^{1/3} = -0 + C \rightarrow C = 12$$

$$G(t) = 27 + \left(\frac{-t+12}{3}\right)^3$$

$G(3) = 54$ degrees celcius.

Problem 5

a)

$$f(x) = \frac{3}{2x^2 - 7x + 5} \rightarrow f'(x) = \frac{-3(4x-7)}{(2x^2 - 7x + 5)^2}$$

$$f'(3) = \frac{-3*5}{4} = \frac{-15}{4}$$

b)

We look for time intervals during which the two velocities have the same sign:

$$f'(x) = \frac{-3(4x-7)}{(2x^2 - 7x + 5)^2}$$

$x = \frac{7}{4} \rightarrow f(\frac{7}{4})$ is a local max since the derivative changes sign from positive to negative.

$x = \frac{7 \pm 3}{4} \rightarrow x = \frac{5}{2}, x = 1 \rightarrow$ neither local max nor local min at these two points (no sign change).

Additionally, note that the last two points are vertical asymptotes (not in the domain).

c)

$$\int_5^\infty f(x)\,dx = \int_5^\infty \frac{2}{2x-5} - \frac{1}{x-1}\,dx = \int_5^\infty \frac{2}{2x-5}\,dx - \int_5^\infty \frac{1}{x-1}\,dx =$$

$$= \lim_{b\to\infty}\left(\ln|2x-5| - \ln|x-1|\right)\Big|_5^b = \lim_{b\to\infty}\left(\ln\left|\frac{2x-5}{x-1}\right|\right)\Bigg|_5^b = \ln 2 - \ln(5/4) = \ln(8/5)$$

d)

$$\sum \frac{3}{2n^2 - 7n + 5}$$

The series converges due to the Integral Test. The function that defines the sequence is positive, continuous, and decreasing. This was shown in part a). Since the improper integral converges, it follows that the series converges as well.

Problem 6

a)

$$f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+...=$$

$$=0+1*x+\frac{(-1)(1)}{2!}x^2+\frac{(-2)(-1)}{3!}x^3+\frac{(-3)(2)}{4!}x^4+...=$$

$$=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+...+\frac{(-1)^{n+1}x^n}{n}+...$$

b)

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\rightarrow \text{Alternating Harmonic Series (converges)}$$

The series converges conditionally. In absolute value, this turns into the harmonic series, which diverges.

c)

$$g(0)=0\rightarrow C=0$$

$$g'(x)=f(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n}=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+...+\frac{(-1)^{n+1}x^n}{n}+...$$

$$g(x)=C+\frac{x^2}{2}-\frac{x^3}{3*2}+\frac{x^4}{4*3}-\frac{x^5}{5*4}+...+\frac{(-1)^{n+!}x^{n+1}}{(n+1)*n}+...$$

$$g(x)=\frac{x^2}{2}-\frac{x^3}{3*2}+\frac{x^4}{4*3}-\frac{x^5}{5*4}+...+\frac{(-1)^{n+!}x^{n+1}}{(n+1)*n}+...$$

d)

$$\left|P_4\left(\frac{1}{2}\right)-g\left(\frac{1}{2}\right)\right|<\left|\frac{(1/2)^{4+1}}{(4+1)*4}\right|=\frac{1}{32*20}=\frac{1}{640}<\frac{1}{500}$$