

Problem 1 | 2014 BC (Problem 1 | 2014 AB)

a)

$$A(t) = 6.687 * (0.931)^t$$

$$\text{AvgRate}_{[0,30]} = \frac{A(30) - A(0)}{30 - 0} = -0.196802 \approx -0.197 \text{ pounds per day.}$$

The average rate of change of $A(t)$ on $[0, 30]$ is -0.197 pounds per day.

b)

$$A(t) = 6.687 * (0.931)^t$$

$$A'(15) \approx -0.163591 \approx -0.164 \text{ pounds per day.}$$

At $t = 15$ days, the amount of grass clippings remaining in the bin is decreasing at a rate of 0.164 pounds per day. The rate is negative because the grass clippings decompose.

c)

$$A(t) = 6.687 * (0.931)^t$$

$$\text{Average Amount} = \frac{1}{30 - 0} \int_0^{30} A(t) dt = 2.75264 \text{ pounds [TI-84]}$$

Solve:

$$A(t) = 2.75264$$

$$t = 12.4147 \text{ days}$$

The amount of grass clippings in the bin is equal to the average amount of grass clippings over the interval $[0, 30]$ when $t = 12.415$ days.

d)

$$A(t) = 6.687 * (0.931)^t$$

$$L(t) = A(30) + A'(30)(t - 30) = 0.782928 - 0.0559762(t - 30)$$

Solve:

$$L(t) = \frac{1}{2}$$

$$t = 35.0544 \text{ days [TI-84]}$$

According to the linear approximation, we predict that at approximately $t = 35.054$ days there will be half a pound of grass left in the bin. Note that this time value is an underestimate, since we are using a tangent line to a graph that is concave up (exponential decay).

Problem 2 | 2014 BC

a)

$$\text{Area} = \frac{\pi * 3^2}{4} + \int_0^{\pi/2} \frac{1}{2} (3 - 2 \sin(2\theta))^2 d\theta \approx 9.70796$$

The area of the shaded region R is approximately 9.708 square units.

b)

$$x(\theta) = r(\theta) \cos \theta = (3 - 2 \sin(2\theta)) \cos \theta$$

$$\frac{dx}{d\theta} \Big|_{\theta=\frac{\pi}{6}} \approx -2.36603 \text{ [TI-84]}$$

The rate of change of x with respect to the angle θ is approximately -2.366 when $\theta = \pi / 6$.

c)

$$D(\theta) = 3 - (3 - 2 \sin(2\theta)) = 2 \sin(2\theta)$$

$$D'(\pi / 3) = 4 \cos(2\theta) \Big|_{\theta=\pi/3} = -2$$

The distance between the two polar curves is decreasing at a rate of 2.000 units per radian, when the angle is $\pi / 3$.

d)

$$r(\theta) = 3 - 2 \sin(2\theta)$$

$$\frac{dr}{dt} = \frac{dr}{d\theta} \frac{d\theta}{dt}$$

$$\frac{dr}{dt} \Big|_{\theta=\frac{\pi}{6}} = (-4 \cos(2\theta)) \Big|_{\theta=\frac{\pi}{6}} * 3 = (-2) * 3 = -6$$

When $\theta = \pi / 6$, the distance from the curve $r(\theta)$ to the pole is decreasing at a rate of 6.000 units of length per unit of time.

Problem 3 | 2014 BC (Problem 3 | 2014 AB)

a)

$$g(3) = \int_{-3}^3 f(t) dt = \frac{3*4}{2} + \frac{2*4}{2} + \frac{1*(-2)}{2} = 6 + 4 - 1 = 9$$

b)

By the Fundamental Theorem of Calculus, the first and second derivatives of $g(x)$ are:

$$g'(x) = f(x) > 0$$

$$g''(x) = f'(x) < 0$$

$$(-5, -3) \text{ and } (0, 2)$$

These are the only intervals for which f is both positive and decreasing (or where g is both increasing and concave down.)

c)

$$h(x) = \frac{g(x)}{5x}$$

$$h'(x) = \frac{5x g'(x) - 5g(x)}{25x^2}$$

$$h'(3) = \frac{5*3*g'(3) - 5g(3)}{25*9} = \frac{15*f(3) - 5*9}{225} = \frac{15*(-2) - 45}{225} = \frac{-3}{9} = -\frac{1}{3}$$

d)

$$p(x) = f(x^2 - x)$$

$$p'(x) = f'(x^2 - x) * (2x - 1)$$

$$p'(-1) = f'(2) * (-3) = (-2)(-3) = 6$$

Problem 4 | 2014 BC (Problem 4 | 2014 AB)

a)

$$\text{Average Acceleration}_{\text{TRAIN A}[2,8]} = \frac{v(8) - v(2)}{8 - 2} = \frac{-120 - 100}{8 - 2} = \frac{-220}{6} = \frac{-110}{3} \frac{\text{meters}}{\text{min}^2}$$

b)

If the velocity function of train A is differentiable, then it is also continuous in the given interval. We can hence apply the Intermediate Value Theorem on $[5, 8]$ to conclude that the velocity function must take on every value between 40 and -120 at least once; this interval also includes the value of -100 meters per minute, so the train's velocity must have been -100 meters/minute at least once on the interval $(5, 8)$.

c)

$$x(t) - x(2) = \int_{w=2}^{w=t} v_A(w) dw$$

$$x(t) - 300 = \int_{w=2}^{w=t} v_A(w) dw$$

$$x(t) = 300 + \int_{w=2}^{w=t} v_A(w) dw$$

$$\int_{w=2}^{w=t} v_A(w) dw \approx \frac{(5-2)(100+40)}{2} + \frac{(8-5)(-120+40)}{2} + \frac{(12-8)(-150-120)}{2} =$$

$$\int_{w=2}^{w=t} v_A(w) dw \approx 210 - 120 - 540 = -450 \text{ meters.}$$

$$x(12) \approx 300 - 450 = -150 \text{ meters.}$$

d)

Pythagorean Theorem:

$$z^2(t) = x^2(t) + y^2(t)$$

$$(x, y, z) = (300, 400, 500)$$

$$2z(t)z'(t) = 2x(t)x'(t) + 2y(t)y'(t)$$

$$y'(2) = v_B(2) = 125$$

$$x'(2) = v_A(2) = 100$$

$$z'(2) = \frac{x(2)x'(2) + y(2)y'(2)}{z(2)} = \frac{300 * 100 + 400 * 125}{500} = 160 \text{ meters / min}$$

The rate at which the distance between train A and train B is changing at time $t = 2$ minutes is 160 meters per minute.

Problem 5 | 2014 BC

a)

$$Area = \int_0^1 y_{above} - y_{below} dx = \int_0^1 xe^{x^2} - (-2x) dx = \left(\frac{1}{2} e^{x^2} + x^2 \right) \Big|_{0,1} = \frac{e}{2} + 1 - \frac{1}{2} - 0 = \frac{e+1}{2}$$

b)

Washer Method:

$$R(x) = 2 + xe^{x^2}$$

$$r(x) = 2 + (-2x) = 2 - 2x$$

$$V = \pi \int_0^1 (2 + xe^{x^2})^2 - (2 - 2x)^2 dx$$

c)

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx + \sqrt{1^2 + (-2)^2} + (1 * e^{1^2} - (-2 * 1))$$

$$L = \int_0^1 \sqrt{1 + (e^{x^2} + 2x^2 e^{x^2})^2} dx + \sqrt{5} + e + 2$$

CalculusQuestions.org

Problem 6 | 2014 BC

a)

We apply the Ratio Test:

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{2^{n+1}(x-1)^{n+1}}{n+1} * \frac{n}{2^n(x-1)^n} \right| = \left| \frac{2(x-1)}{1} * \frac{n}{n+1} \right| \rightarrow 2|x-1| < 1$$

$$|x-1| < \frac{1}{2}$$

$$R = \frac{1}{2}$$

b)

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n} (x-1)^n = \frac{2}{1}(x-1) - \frac{2^2}{2}(x-1)^2 + \frac{2^3}{3}(x-1)^3 - \frac{2^4}{4}(x-1)^4 + \dots$$

$$f'(x) = \sum_{n=1}^{\infty} (-1)^{n+1} 2^n (x-1)^{n-1}$$

$$f'(x) = 2 - 2^2(x-1) + 2^3(x-1)^2 - 2^4(x-1)^3 + \dots + (-1)^{n+1} 2^n (x-1)^{n-1} + \dots$$

$$f'(x) = 2 - 4(x-1) + 8(x-1)^2 - 16(x-1)^3 + \dots + (-1)^{n+1} 2^n (x-1)^{n-1} + \dots$$

c)

$$a = 2$$

$$r = -2(x-1)$$

$$f'(x) = \frac{a}{1-r} = \frac{2}{1-(-2)(x-1)} = \frac{2}{1+2x-2} = \frac{2}{2x-1}$$

$$f(x) = \int \frac{2}{2x-1} dx = C + \ln|2x-1|$$

$$f(1) = 0$$

$$C = 0$$

$$f(x) = \ln|2x-1|$$