Problem 1 | 2014 AB

a)

$A(t) = 6.687 * (0.931)^t$

$AvgRate_{[0,30]} = \dfrac{A(30) - A(0)}{30 - 0} = -0.196802 \approx -0.197$ pounds per day.

The average rate of change of $A(t)$ on [0, 30] is -0.197 pounds per day.

b)

$A(t) = 6.687 * (0.931)^t$

$A'(15) \approx -0.163591 \approx -0.164$ pounds per day.

At $t = 15$ days, the amount of grass clippings remaining in the bin is decreasing at a rate of 0.164 pounds per day. The rate is negative because the grass clippings decompose.

c)

$A(t) = 6.687 * (0.931)^t$

Average Amount $= \dfrac{1}{30 - 0} \int_0^{30} A(t)\,dt = 2.75264$ pounds [TI-84]

Solve:

$A(t) = 2.75264$

$t = 12.4147$ days

The amount of grass clippings in the bin is equal to the average amount of grass clippings over the interval [0, 30] when $t = 12.415$ days.

d)

$A(t) = 6.687 * (0.931)^t$

$L(t) = A(30) + A'(30)(t - 30) = 0.782928 - 0.0559762(t - 30)$

Solve:

$L(t) = \dfrac{1}{2}$

$t = 35.0544$ days [TI-84]

According to the linear approximation, we predict that at approximately $t = 35.054$ days there will be half a pound of grass left in the bin. Note that this time value is an underestimate, since we are using a tangent line to a graph that is concave up (exponential decay).

Problem 2 | 2014 AB

a)

Intersection points:

$x = 0, x = 2.3$

$$V_{Washer} = \pi \int_{x=0}^{x=2.3} R^2(x) - r^2(x)\,dx = \pi \int_{x=0}^{x=2.3} (2+4)^2 - (2 + x^4 - 2.3x^3 + 4)\,dx$$

$V_{Washer} = 98.86779 \approx 98.868$ cubic units.

b)

The leg of the isosceles right triangle is given by:

$$m = y_{above} - y_{below} = 4 - (x^4 - 2.3x^3 + 4) = 2.3x^3 - x^4$$

$$A(x) = \frac{base * height}{2} = \frac{m^2}{2} = \frac{\left(2.3x^3 - x^4\right)^2}{2}$$

$$V = \int_0^{2.3} A(x)\,dx = \int_0^{2.3} \frac{\left(2.3x^3 - x^4\right)^2}{2}\,dx = 3.57372 \approx 3.574 \text{ cubic units.}$$

c)

$$2 * \int_0^k 4 - (x^4 - 2.3x^3 + 4)\,dx = \int_0^{2.3} 4 - (x^4 - 2.3x^3 + 4)\,dx$$

$$2 * \int_0^k -x^4 + 2.3x^3\,dx = \int_0^{2.3} -x^4 + 2.3x^3\,dx$$

Problem 3 | 2014 AB

a)

$$g(3) = \int_{-3}^{3} f(t)\,dt = \frac{3*4}{2} + \frac{2*4}{2} + \frac{1*(-2)}{2} = 6 + 4 - 1 = 9$$

b)

By the Fundamental Theorem of Calculus, the first and second derivatives of $g(x)$ are:

$g'(x) = f(x) > 0$

$g''(x) = f'(x) < 0$

$(-5, -3)$ and $(0, 2)$

These are the only intervals for which f is both positive and decreasing (or where g is both increasing and concave down.)

c)

$$h(x) = \frac{g(x)}{5x}$$

$$h'(x) = \frac{5x\,g'(x) - 5g(x)}{25x^2}$$

$$h'(3) = \frac{5*3*g'(3) - 5g(3)}{25*9} = \frac{15*f(3) - 5*9}{225} = \frac{15*(-2) - 45}{225} = \frac{-3}{9} = -\frac{1}{3}$$

d)

$p(x) = f(x^2 - x)$

$p'(x) = f'(x^2 - x)*(2x - 1)$

$p'(-1) = f'(2)*(-3) = (-2)(-3) = 6$

Problem 4 | 2014 AB

a)

$$\text{Average Acceleration}_{TRAIN\ A\,[2,8]} = \frac{v(8)-v(2)}{8-2} = \frac{-120-100}{8-2} = \frac{-220}{6} = \frac{-110}{3}\frac{meters}{\min^2}$$

b)

If the velocity function of train A is differentiable, then it is also continuous in the given interval. We can hence apply the Intermediate Value Theorem on [5, 8] to conclude that the velocity function must take on every value between 40 and -120 at least once; this interval also includes the value of -100 meters per minute, so the train's velocity must have been -100 meters/minute at least once on the interval $(5,8)$.

c)

$$x(t)-x(2)=\int_{w=2}^{w=t} v_A(w)\,dw$$

$$x(t)-300=\int_{w=2}^{w=t} v_A(w)\,dw$$

$$x(t)=300+\int_{w=2}^{w=t} v_A(w)\,dw$$

$$\int_{w=2}^{w=t} v_A(w)\,dw \approx \frac{(5-2)(100+40)}{2}+\frac{(8-5)(-120+40)}{2}+\frac{(12-8)(-150-120)}{2}=$$

$$\int_{w=2}^{w=t} v_A(w)\,dw \approx 210-120-540=-450\ meters.$$

$$x(12)\approx 300-450=-150\ meters.$$

d)

Pythagorean Theorem:

$$z^2(t)=x^2(t)+y^2(t)$$

$$(x,y,z)=(300,400,500)$$

$$2z(t)z'(t)=2x(t)x'(t)+2y(t)y'(t)$$

$$y'(2)=v_B(2)=125$$

$$x'(2)=v_A(2)=100$$

$$z'(2)=\frac{x(2)x'(2)+y(2)y'(2)}{z(2)}=\frac{300*100+400*125}{500}=160\ meters/\min$$

The rate at which the distance between train A and train B is changing at time $t=2$ minutes is 160 meters per minute.

Problem 5 | 2014 AB

a)
$f(x)$ has a relative minimum at $x=1$ because its first derivative changes sign from negative to positive there, and the function is differentiable (and hence continuous) there.

b)
We apply Rolle's Theorem (or MVT) for $f'(x)$ on [-1, 1]:

$f'(-1)=0=f'(1)$

$f'(x)$ is continuous one [-1, 1] and differentiable on (-1 , 1)

$\rightarrow$ Therefore, $f''(c)=0$ at least once in (-1 , 1).

c)

$h(x)=\ln\left(f(x)\right)$

$h'(x)=\dfrac{1}{f(x)}f'(x)=\dfrac{f'(x)}{f(x)}$

$h'(3)=\dfrac{f'(3)}{f(3)}=\dfrac{0.5}{7}=\dfrac{1}{14}$

d)

$\int_{-2}^{3} f'\left(g(x)\right)g'(x)\,dx$

$u=g(x)$

$du=g'(x)\,dx$

$g(-2)=-1$

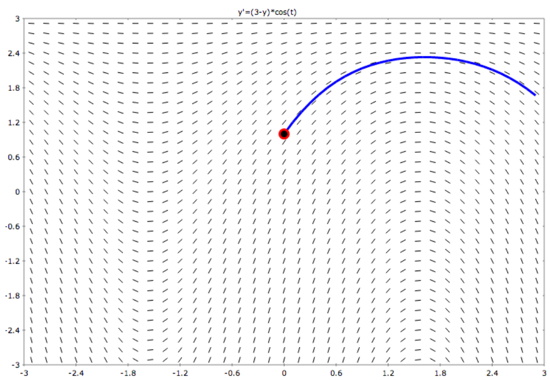
$g(3)=1$

$\int_{-2}^{3} f'\left(g(x)\right)g'(x)\,dx=\int_{-1}^{1} f'\left(u\right)du=f(1)-f(-1)=2-8=-6$

Problem 6 | 2014 AB

a)

*Note the solution below should continue into the second quadrant, since the function is defined for all real numbers. Shown in the slope field is only the first quadrant solution.



b)

$$\frac{dy}{dx} = (3-y)\cos x$$

$$x = 0, y = 1 \rightarrow \frac{dy}{dx} = slope = 2\cos 0 = 2$$

$$y - 1 = 2(x - 0) \rightarrow y = 2x + 1$$

$$f(0.2) \approx 2 * 0.2 + 1 = 1.4$$

c)

$$\frac{dy}{dx} = (3-y)\cos x$$

$$\frac{1}{3-y}dy = \cos x\,dx$$

$$\int \frac{1}{3-y}dy = \int \cos x\,dx$$

$$-\ln|3-y| = \sin x + C$$

$$-\ln 2 = 0 + C \rightarrow C = -\ln 2$$

$$\ln|3-y| = -\sin x + \ln 2$$

$$3 - y = e^{-\sin x}e^{\ln 2} = 2e^{-\sin x}$$

$$y = f(x) = 3 - 2e^{-\sin x}$$