Problem 1

a)

$$W'(12) \approx \frac{W(15)-W(9)}{15-9} = \frac{67.9-61.8}{6} = \frac{6.1}{6} \approx 1.017 \frac{\text{degrees Fahrenheit}}{\text{minute}}$$

At $t = 12\text{minutes}$, we estimate that the temperature of water is increasing at a rate approximately equal to the average rate of change between 9 and 15 minutes, which is 1.017 degrees Fahrenheit per minute. (Answer rounded to three decimal places.)

b)

$\int_0^{20} W'(t)dt = W(20) - W(0) = 71.0 - 55.0 = 16 \text{ degrees Fahrenheit}$. This is the Net Change in the temperature of water from 0 to 20 minutes.

c)

$$L_4 = W(0)(44-0) + W(4)(9-5) + W(9)(15-9) + W(15)(20-15) =$$
$$L_4 = 55*4 + 57.1*5 + 61.8*6 + 67.9*5 = 1215.8$$
$$\frac{1}{20}\int_0^{20} W(t)dt \approx \frac{1}{20} L_4 = \frac{1215.8}{20} = 60.790 \text{ degrees Fahrenheit}$$

The approximate average temperature over the 20 minutes is 60.790 degrees Fahrenheit.

This is an underestimate because the function used as a height of the rectangles in the Riemann Sum is increasing; hence the resulting rectangles fail to capture the entire area under the graph.

d)

$$W(25) - W(20) = \int_{20}^{25} W'(t)dt = 2.04315\ (TI-84)$$
$$W(25) = W(20) + 2.04315 = 71.0 + 2.04315 = 73.043 \text{ deg Fahrenheit}$$

Problem 2

a)

Intersection Point x-coordinate = 3.69344 = a

$$Area = \int_1^a \ln x\, dx + \int_a^5 5 - x\, dx = 2.13225 + 0.85355 = 2.9858 \approx 2.986$$

OR

$$Area = \int_{y=0}^{y=5-a} X_{FAR} - X_{NEAR}\, dy = \int_{y=0}^{y=1.30656} (5-y) - e^y\, dy = 2.9858 \approx 2.986$$

b)

The side of a typical square is given by the y-value.

$$V = \int_1^5 A(x)dx \text{, where } A(x) = \begin{cases} (\ln x)^2 & \text{when } 1 \le x \le 3.69344 \\ (5-x)^2 & \text{when } 3.69344 \le x \le 5 \end{cases}$$

c)

The horizontal line $y = k$ will intersect the graphs of $y = \ln x$ and $y = 5 - x$ at the points (b, k) and (c, k), respectively. We can express the two areas as infinite sums of rectangles in the vertical direction.

$X_{FAR} = 5 - y \Leftarrow y = 5 - x$

$X_{NEAR} = e^y \Leftarrow y = \ln x$

Note that the value of $x = a$ was determined in part a).

$$\int_{y=0}^{y=k} X_{FAR} - X_{NEAR}\, dy = \int_{y=k}^{y=5-a} X_{FAR} - X_{NEAR}\, dy$$

$$\int_{y=0}^{y=k} 5 - y - e^y\, dy = \int_{y=k}^{y=1.30656} 5 - y - e^y\, dy$$

Problem 3

a)

$$g(2)=\int_1^2 f(t)dt=\tfrac{1}{2}*base*height=\tfrac{1}{2}*1*\tfrac{-1}{2}=\tfrac{-1}{4}$$

$$g(-2)=\int_1^{-2} f(t)dt=-\int_{-2}^{1} f(t)dt=-[(right\ triangle)-(semicircle)]=-[\tfrac{3}{2}-\tfrac{\pi}{2}]=\tfrac{\pi-3}{2}$$

Note that the minus sign in front of 'semicircle' is added because the semi-circle lies below the horizontal axis, whereas the right triangle lies above.

b)

$g'(x)=f(x), g''(x)=f'(x)$

$g'(-3)=f(-3)=2$

$g''(-3)=f'(-3)=1$ (read slope from the graph)

c)

$g'(x)=0\rightarrow f(x)=0\rightarrow x=-1,1$

$x=-1\rightarrow g'(x)$, which is $f(x)$, changes from positive to negative

$g(-1)$relative maximum

x=1 $\rightarrow g'(x)$, which is $f(x)$, does not change sign,but $g''(x)$ does.

$g(1)$not a relative extremum. It is an inflection point.

d)

An inflection point occurs where $g(x)$ is continuous and $g''(x)$ changes sign but $g'(x)$ does not.

In the context of this problem, we are looking for points where $f'(x)$ changes sign, yet $f(x)$ maintains its sign. This occurs at (-2, 3), (0, -1), and (1, 0). These are the three inflection points.

Problem 4

a)

$$f'(x) = \frac{-2x}{2\sqrt{25-x^2}} = \frac{-x}{\sqrt{25-x^2}} [\textit{Chain Rule}]$$

Note that $\frac{dy}{dx} = \frac{-x}{y}$ [Remember this from implicit differentiation?]

b)

$$x = -3 \to y = \sqrt{25-9} = 4$$
$$(-3, 4)$$
$$\left.\frac{dy}{dx}\right|_{(-3,4)} = \frac{-x}{y} = \frac{3}{4} \to y - 4 = \frac{3}{4}(x+3) \to 4y - 3x = 25$$

c)

$g(x)$ will be continuous at $x = -3$ if and only if $\lim_{x\to -3} g(x) = g(-3)$

We investigate the one-sided limits of the piece-wise function:

$$\lim_{x\to -3^-} g(x) = \lim_{x\to -3^-} f(x) = 4 = g(-3)$$
$$\lim_{x\to -3^+} g(x) = \lim_{x\to -3^+} (x+7) = 4$$

The one-sided limits coincide and are also equal to the y-value, therefore the function is continuous at –3.

d)

$$u = 25 - x^2$$
$$du = -2x\,dx$$
$$\int_0^5 x\sqrt{25-x^2}\,dx = \frac{-1}{2}\int_0^5 -2x\sqrt{25-x^2}\,dx = \frac{-1}{2}\int_{u(0)}^{u(5)} \sqrt{u}\,du =$$
$$= \frac{-1}{2}\int_{25}^{0} \sqrt{u}\,du = \frac{1}{2}\int_0^{25} \sqrt{u}\,du = \frac{1}{2}\frac{2}{3}u^{\frac{3}{2}}\Big|_0^{25} = \frac{1}{3}(125-0) = \frac{125}{3}$$

Problem 5

a)

$\frac{dB}{dt}\Big|_{B=40} = \frac{1}{5}(100-40) = 12\ grams/day$

$\frac{dB}{dt}\Big|_{B=70} = \frac{1}{5}(100-70) = 6\ grams/day$

The bird is gaining weight faster when it weighs 40 grams.

b)

$\frac{d^2B}{dt^2} = \frac{d}{dt}\left(\frac{dB}{dt}\right) = \frac{d}{dt}\left(\frac{1}{5}(100-B)\right) = \frac{-1}{5}\frac{dB}{dt} = \frac{-1}{5}\frac{1}{5}(100-B) = \frac{-1}{25}(100-B)$

When B is less than 100, the second derivative suggests that the graph must be concave down for all values of B in this range. The given graph is concave up at first, and then concave down.

c)

$\frac{dB}{dt} = \frac{1}{5}(100-B)$

$\frac{1}{100-B}dB = \frac{1}{5}dt$

$\int \frac{1}{100-B}dB = \int \frac{1}{5}dt$

$-\int \frac{-1}{100-B}dB = \int \frac{1}{5}dt$

$-\ln|100-B| = \frac{t}{5} + C$

$B(0) = 20 \rightarrow -\ln 80 = C$

$-\ln|100-B| = 0.2t - \ln 80$

$\ln|100-B| = \ln 80 - 0.2t$

$100 - B = e^{\ln 80 - 0.2t}$

$B(t) = 100 - 80e^{-0.2t}$

Problem 6

a)

The particle will be moving to the left when the velocity function is negative. So we look for:

$$v(t)=\cos\left(\frac{\pi}{6}t\right)<0 \text{ for } 0\le t\le 12 \to \frac{\pi}{2}<\frac{\pi}{6}t<\frac{3\pi}{2}\to 3<t<9$$

The particle will be moving to the left for $t-$values between 3 and 9.

b)

$$\text{Total Distance on [0, 6]} = \int_0^6 |v(t)|\,dt = \int_0^6 \left|\cos\left(\frac{\pi}{6}t\right)\right| dt$$

c)

$$a(t)=v'(t)=-\sin\left(\frac{\pi}{6}t\right)\frac{\pi}{6} \qquad a(4)=-\sin\left(\frac{\pi}{6}4\right)\frac{\pi}{6}=\frac{-\pi}{6}\frac{\sqrt{3}}{2}=\frac{-\sqrt{3}\pi}{12}$$

Speed is the absolute value of velocity. We know, from part a), that the velocity at $t=4$ is negative, which means

$$speed = w(t)=|v(t)|=\begin{cases} v(t) & when\ v(t)\ge 0 \\ -v(t) & when\ v(t)<0 \end{cases}$$

$$w'(4)=-v'(4)=-a(4)=\frac{\sqrt{3}\pi}{12}>0$$

We conclude that the speed of the particle at $t=4$ is increasing.

d)

$$FTC: x(4)-x(0)=\int_0^4 v(t)dt \Rightarrow x(4)=x(0)+\int_0^4 \cos\left(\frac{\pi}{6}t\right)dt=-2+\frac{6}{\pi}\int_0^4 \frac{\pi}{6}\cos\left(\frac{\pi}{6}t\right)dt=$$

$$-2+\frac{6}{\pi}\sin\left(\frac{\pi}{6}t\right)\Big|_0^4=-2+\frac{6}{\pi}(\sin\frac{2\pi}{3}-\sin 0)=-2+\frac{6}{\pi}\frac{\sqrt{3}}{2}=\left(\frac{3\sqrt{3}}{\pi}-2\right) units$$