AP Calculus AB/BC Free Response Questions – 2011 SOLUTIONS [draft]

Problem 1 BC [CALCULATOR]

a)

Speed of the particle at time $t=3$

$$\sqrt{\left(x'(3)\right)^2+\left(y'(3)\right)^2}=\sqrt{(4t+1)^2+\left(\sin(t^2)\right)^2}\Big|_{t=3}=\sqrt{169+\left(\sin(9)\right)^2}\approx 13.007$$

Acceleration vector at $t=3$

$$\left\langle x''(3),\, y''(3)\right\rangle=\left\langle 4,\, 2t\cos(t^2)\right\rangle_{t=3}=\left\langle 4,\, 6\cos 9\right\rangle=\left\langle 4,\, -5.467\right\rangle$$

b)

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\sin(t^2)}{4t+1}=\frac{\sin 9}{13}\approx 0.032$$

c)

$$x(3)-x(0)=\int_{t=0}^{t=3}x'(t)\,dt$$

$$x(3)=x(0)+\int_{t=0}^{t=3}4t+1\,dt=0+(2t^2+t)\Big|_0^3=18+3=21$$

$$y(3)-y(0)=\int_{t=0}^{t=3}y'(t)\,dt$$

$$y(3)=y(0)+\int_{t=0}^{t=3}\sin(t^2)\,dt=-4+0.773563\approx -3.266$$

d) Total Distance equals arc length:

$$\int_{t=0}^{t=3}\sqrt{[x'(t)]^2+[y'(t)]^2}\,dt=\int_0^3\sqrt{[4t+1]^2+[\sin(t^2)]^2}\,dt=21.091190452\ldots\approx 21.091$$

Problem 2 BC [CALCULATOR]

a)

$$H'(3.5) \approx \frac{H(5)-H(2)}{5-2} = \frac{52-60}{3} = \frac{-8}{3}\frac{\deg Celsius}{\min}$$

b)

$\frac{1}{10}\int_{t=0}^{t=10} H(t)\,dt$ represents the average temperature of the tea over the ten-minute interval. The units are degree Celsius.

$$\frac{1}{10}\int_{t=0}^{t=10} H(t)\,dt \approx \frac{1}{10}(T_1+T_2+T_3+T_4)$$

$$T_1 = \frac{(66+60)(2-0)}{2} = 126$$

$$T_2 = \frac{(60+52)(5-2)}{2} = 168$$

$$T_3 = \frac{(52+44)(9-5)}{2} = 192$$

$$T_4 = \frac{(44+43)(10-9)}{2} = 43.5$$

$$\frac{1}{10}\int_{t=0}^{t=10} H(t)\,dt \approx \frac{1}{10}(T_1+T_2+T_3+T_4) = \frac{529.5}{10} = 4 = 52.950 \deg Celsius$$

The average temperature over the ten minutes, using trapezoidal approximation with four trapezoids, is 52.950 degrees Celsius.

c)

$$\int_0^{10} H'(t)\,dt = H(10) - H(0) = 43 - 66 = -23 \deg Celsius$$

The net change in temperature over the ten-minute interval is – 23 degrees Celsius. In other words, the temperature dropped 23 degrees.

d)

$$B(0) = 100$$

$$B'(t) = -13.84e^{-0.173t}$$

$$B(10) - B(0) = \int_0^{10} B'(t)\,dt$$

$$B(10) = B(10) + \int_0^{10} B'(t)\,dt = 100 + \int_0^{10} -13.84e^{-0.173t}\,dt = 100 - 65.8172 \approx 34.183$$

$$H(10) = 43$$

$$H(10) - B(10) = 43 - 34.183 = 8.817 \deg Celsius$$

The biscuits are approximately 9 degrees Celsius cooler than the tea.

Problem 3 BC [NO CALCULATOR]

a)

The perimeter is given by:

$$P = 1 + k + e^{2k} + L = 1 + k + e^{2k} + \int_{x=0}^{x=k} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$P = 1 + k + e^{2k} + \int_{x=0}^{x=k} \sqrt{1 + \left(2e^{2x}\right)^2}\, dx = 1 + k + e^{2k} + \int_{x=0}^{x=k} \sqrt{1 + 4e^{4x}}\, dx$$

b)

$$V_{DISK} = \int_{x=0}^{x=k} \pi \left[R(x)\right]^2 dx = \int_{x=0}^{x=k} \pi \left[e^{2x}\right]^2 dx = \pi \int_{x=0}^{x=k} e^{4x} dx$$

$$V_{DISK} = \frac{\pi}{4} \int_{x=0}^{x=k} 4e^{4x} dx = \frac{\pi}{4} e^{4x}\Big|_0^k = \frac{\pi}{4}(e^{4k} - e^0) = \frac{\pi}{4}(e^{4k} - 1)$$

c)

$$V(k(t)) = \frac{\pi}{4}(e^{4k(t)} - 1)$$

$$\frac{dV}{dt} = \frac{\pi}{4} e^{4k(t)} * 4\frac{dk}{dt} = \frac{\pi e^{4k(t)}}{3}\Bigg|_{k=\frac{1}{2}} = \frac{\pi e^2}{3} \frac{units^3}{time\ unit}$$

Problem 4 BC [NO CALCULATOR]

a)

$$g(-3)=2(-3)+\int_0^{-3} f(t)\,dt=-6+\frac{-\pi r^2}{4}=-6-\frac{9\pi}{4}$$

$$g'(x)=2+f(x) \text{ by } FTC \Rightarrow g'(-3)=2+f(-3)=2+0=2$$

b)

Closed Interval Method

Find critical numbers in the open interval (-4, 3):

$g'(x)=2+f(x)=0$ ('undefined' case impossible since f is defined for all x in given interval)

$$\Rightarrow f(x)=-2 \Rightarrow -2x+3=-2 \Rightarrow x=\frac{5}{2}$$

$$g(\tfrac{5}{2})=2*\tfrac{5}{2}+\int_0^{2.5} f(t)\,dt=5+\frac{9}{4}-1=\frac{25}{4}$$

$$g(-4)=2*(-4)+\frac{-\pi 3^2}{4}+\frac{\pi 1^2}{4}=-8-2\pi$$

$$g(3)=2*3+\int_0^{3} f(t)\,dt=6+0=6$$

$g(2.5)$ global maximum using the Closed Interval Method.

c)

At a point of inflection, the original function must be continuous and its second derivative must change sign while the first derivative does *not* change sign.

$$g'(x)=2+f(x)$$

$$g''(x)=f'(x)$$

The only inflection point occurs at (0, 3). The function changes concavity while maintaining its increasing behavior on both sides of the given point.

d)

$$AVG\ RATE\ OF\ f(t)_{[-4,3]}=\frac{f(3)-f(-4)}{3-(-4)}=\frac{-3-(-1)}{7}=\frac{-2}{7}$$

The instantaneous rate of change is never equal to the average rate of change on the given interval. (Geometrically, the slope of the tangent to the graph is never $\frac{-2}{7}$).

This does not contradict the Mean Value Theorem because the function is not differentiable on the open interval. Differentiability fails at t = −3 (vertical tangent) and t = 0 (corner).

Problem 5 BC [NO CALCULATOR]

a)

$L(t)=W(0)+W'(0)(t-0)$

$L(t)=1400+\frac{1400-300}{25}t=1400+\frac{1100}{25}t=1400+44t$

$f(0.25)\approx L(0.25)=1400+44*\frac{1}{4}=1400+11=1411$ tons of waste are approximately present

b)

$\frac{d^2W}{dt^2}=\frac{1}{25}W'=\frac{1}{25}\frac{dW}{dt}=\frac{1}{25}\left(\frac{1}{25}(W-300)\right)=\frac{W-300}{625}$

$@W(0)=1400:$

$\frac{d^2W}{dt^2}=\frac{1400-300}{625}>0\Rightarrow CONCAVE\ UP\Rightarrow W(0.25)\ IS\ AN\ UNDERESTIMATE$

c)

Separation of Variables

$\frac{dW}{dt}=\frac{1}{25}(W-300)\quad W(0)=1400$

$\frac{1}{W-300}dW=\frac{1}{25}dt$

$\int\frac{1}{W-300}dW=\int\frac{1}{25}dt$

$\ln|W-300|=\frac{1}{25}t+C$

$\ln|1400-300|=\ln 1100=C$

$\ln|W-300|=\frac{1}{25}t+\ln 1100$

$W-300=e^{\frac{1}{25}t+\ln 1100}$

$W(t)=300+1100e^{\frac{1}{25}t}$

Problem 6 BC [NO CALCULATOR]

a)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \ldots$$

b)

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$f(x) = \sin(x^2) + \cos x = \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \ldots\right) + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right)$$

$$f(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \left(\frac{1}{3!} + \frac{1}{6!}\right)x^6 + \ldots = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{121x^6}{720} + \ldots$$

c)

$$\frac{f^{(6)}(0)}{6!} = \frac{-121}{720} \Rightarrow f^{(6)}(0) = -121$$

d)

$$Error\ Bound = \left|\frac{f^{(5)}(k)x^5}{5!}\right| \quad 0 \le k \le \frac{1}{4}$$

$f^{(5)}(k) < 40$ (from the graph)

x=$\frac{1}{4}$

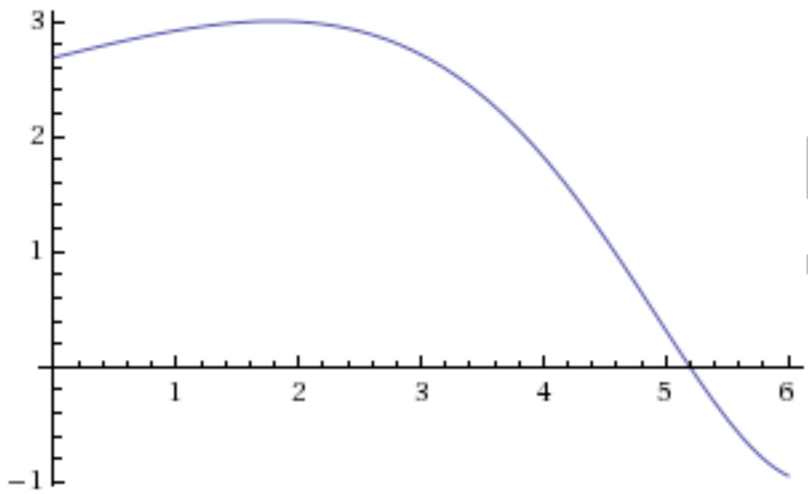
$$Error\ Bound = \left|\frac{f^{(5)}(k)x^5}{5!}\right| = \left|\frac{40\left(\frac{1}{4}\right)^5}{5!}\right| = \frac{40}{4^5 * 5!} = \frac{5*4*2}{4^5(5*4*3*2)} = \frac{1}{3*4^5} = \frac{1}{3072} < \frac{1}{3000}$$

Problem 1 AB [CALCULATOR]

a)

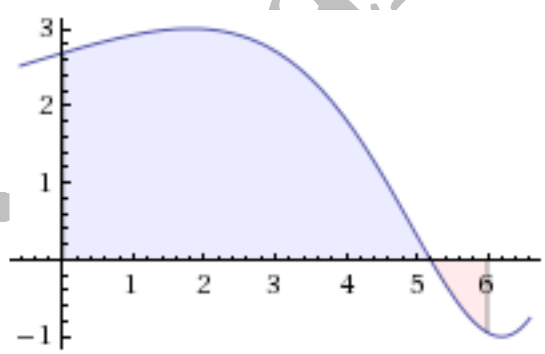
$speed = |v(t)| = |2\sin(e^{t/4}) + 1|$

Given below is a graph of velocity. At *t=5.5*, velocity is decreasing. However, the graph of the speed function would be a reflection about the horizontal axis. This means that speed is increasing at *t=5.5*. One could easily find the slope to the speed function using a graphing device.



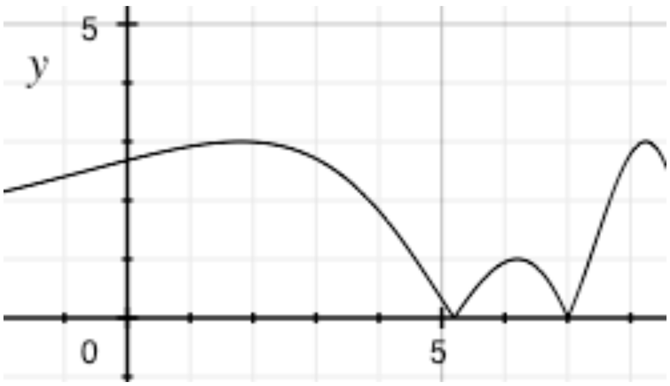
b)

$$v_{avg[0,6]} = \frac{1}{6-0}\int_0^6 v(t)\,dt = \frac{1}{6} * 11.6963 \approx 1.949 \frac{\text{distance units}}{\text{time unit}}$$



c)

$$\text{Total Distance}_{[0,6]} = \int_0^6 |v(t)|\,dt = \int_0^6 \left|2\sin\left(e^{t/4}\right)+1\right|dt = 12.573$$



d)
The particle changes direction when the velocity changes sign. On the given interval this occurs at t = 5.196. (The zero was found use the graphing calculator.)

The position at this time can be found using the Fundamental Theorem of Calculus:

$$x(5.1955) - x(0) = \int_0^{5.1955} v(t)\,dt$$
$$x(5.1955) = x(0) + 12.1348 = 2 + 12.1348 \approx 14.135 \text{ units}$$

Problem 2 AB [CALCULATOR]

Please see solution to Problem 2 BC [CALCULATOR]

Problem 3 AB [NO CALCULATOR]

a)

$f'(x) = 24x^2$

$f'(0.5) = 24 * 0.25 = 6$

$f(0.5) = 1$

$L(x) = f(0.5) + f'(0.5)(x - 0.5) = 1 + 6(x - \frac{1}{2}) = 6x - 2$

$y = 6x - 2$

b)

$$A = \int_{x=0}^{x=\frac{1}{2}} g(x) - f(x)\,dx = \int_{x=0}^{x=\frac{1}{2}} \sin(\pi x) - 8x^3\,dx = \int_{x=0}^{x=\frac{1}{2}} \sin(\pi x)\,dx - \int_{x=0}^{x=\frac{1}{2}} 8x^3\,dx$$

$$A = \frac{-1}{\pi}\int_{x=0}^{x=\frac{1}{2}} -\pi\sin(\pi x)\,dx - \int_{x=0}^{x=\frac{1}{2}} 8x^3\,dx = \left(\frac{-\cos(\pi x)}{\pi} - 2x^4\right)\Bigg|_{x=0}^{x=\frac{1}{2}} = -\frac{1}{8} - \left(\frac{-1}{\pi}\right) = -\frac{1}{8} + \frac{1}{\pi} \text{ square units}$$

c)

Washer Method

$r(x) = 1 - g(x) = 1 - \sin(\pi x)$ (inner radius "near")

$R(x) = 1 - f(x) = 1 - 8x^3$ (outer radius "far")

$[a,b] = [0, \frac{1}{2}]$

$$V = \pi\int_{x=0}^{x=\frac{1}{2}} \left[R(x)\right]^2 - \left[r(x)\right]^2 dx = \pi\int_{x=0}^{x=\frac{1}{2}} \left[1 - 8x^3\right]^2 - \left[1 - \sin(\pi x)\right]^2 dx$$

Problem 4 AB [NO CALCULATOR]

Please see solution to Problem 4 BC [NO CALCULATOR]

Problem 5 AB [NO CALCULATOR]

Please see solution to Problem 5 BC [NO CALCULATOR]

Problem 6 AB [NO CALCULATOR]

a)

$f(0) = 1 - 2\sin 0 = 1$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 1 - 2\sin x = 1$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{-4x} = e^0 = 1$

$\lim_{x\to 0} f(x) = f(0) \Rightarrow \text{continuous at } x = 0$

b)

$$f'(x) = \begin{cases} -2\cos x & x < 0 \\ -4e^{-4x} & x > 0 \end{cases}$$

$f'(x) = -3$

It cannot be to the left of *x=0* since the derivative is less than 2 in absolute value.

$-4e^{-4x} = -3$

$e^{-4x} = \frac{3}{4}$

$-4x = \ln(0.75)$

$x = \frac{\ln(\frac{3}{4})}{-4} = \frac{\ln(\frac{4}{3})}{4}$

c)

$$f_{AVG[-1,1]} = \frac{1}{1-(-1)}\int_{-1}^{1} f(x)\,dx = \frac{1}{2}\left[\int_{-1}^{0} 1 - 2\sin x\,dx + \int_{0}^{1} e^{-4x}\,dx\right]$$

$$f_{AVG[-1,1]} = \frac{1}{2}\left[\left(x + 2\cos x\right)\Big|_{-1}^{0} - \frac{1}{4}e^{-4x}\Big|_{0}^{1}\right] = \frac{1}{2}\left[2 - (-1 + 2\cos(-1) - \frac{1}{4}(e^{-4} - 1)\right] = \frac{1}{2}\left[3 - 2\cos(-1) + \frac{1}{4}(1 - e^{-4})\right]$$

$$f_{AVG[-1,1]} = \frac{1}{2} * \frac{1}{4}\left[12 - 8\cos(-1) + (1 - e^{-4})\right] = \frac{1}{8}\left[13 - 8\cos(-1) - \frac{1}{e^4}\right]$$